Conformal geodesics and geometry of the 3rd order ODEs systems

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The problem

Characterize the geometry of the 3rd order ODEs system

$$y_i'''(x) = f_i(y_j''(x), y_k'(x), y_l(x), x), \quad 1 \le i, j, k, l \le m. \qquad (1)$$

under point transformations. The number of equations will be at most 2.

Historical remark

- Single ODE of 3rd order was studied by S.Chern
- System of 2nd order ODE was studied by M.Fels, D.Grossman

Order / Number	2	3
1	E.Cartan	S.Chern
≥ 2	M.Fels, D.Grossman	???

ODE as a Surface in Jet Space

Consider $J^3(M)$ be a 3 order jet space of the curves on M.
Let's use the following coordinates on the jet space $J^3(M)$:

$$x, y_1, ..., y_m, p_1 = y_1', ..., p_m = y_m', q_1 = y_1'', q_m = y_m'', y_1''', ..., y_m'''.$$

We can represent equation as surface $\mathcal{E}$ in the $J^3(M)$. In coordinates we have the following equations that define $\mathcal{E}$:

$$\mathcal{E} = \{y_i''' = f_i(q_j, p_k, y_l, x)\}. \tag{2}$$

The lifts of the solutions of $\mathcal{E}$ are integral curves of 1-dimensional distribution $E = T\mathcal{E} \cap C(J^3(M))$

$$E = \langle \frac{\partial}{\partial x} + p_i \frac{\partial}{\partial y_i} + q_i \frac{\partial}{\partial p_i} + f^i \frac{\partial}{\partial q_i} \rangle;$$

Filtered Manifolds

- We call a manifold M a filtered manifold if it is supplied with a filtration C^{-i} of the tangent bundle:

$$TM = C^{-k} \supset C^{-k+1} \supset \cdots \supset C^{-1} \supset C^{0} = 0 \qquad (3)$$

where $[C^{-i}, C^{-1}] \subset C^{-i-1}$.

- With every filtered manifold we associate a filtration gr TM of the following form:

$$\operatorname{gr} TM = \bigoplus_{i=1}^{m} \operatorname{gr}_{-i} TM, \qquad (4)$$

where $\operatorname{gr}_{-i} TM = C^{-i}/C^{-i+1}$.

- Space gr TM has a nilpotent Lie algebra structure.
- Let $\mathfrak{m}$ be a nilpotent Lie algebra. Filtered manifold has type $\mathfrak{m}$, if for every point p Lie algebra gr T_pM is eqaual to $\mathfrak{m}$.

Filtered Manifolds

Distribution C^{-1} for the manifold $\mathcal{E}$ has two components:

- 1-dimensial distribution

$$E = \left\langle \frac{\partial}{\partial x} + p_i \frac{\partial}{\partial y_i} + q_i \frac{\partial}{\partial p_i} + f^i \frac{\partial}{\partial q_i} \right\rangle;$$

- m-dimensional vertical distribution

$$V = \left\langle \frac{\partial}{\partial q_i} \right\rangle.$$

A filtered manifold associated with system (1) is

$$C^{-1} \subset C^{-2} \subset C^{-3} = T\mathcal{E}.$$

$$C^{-2} = C^{-1} \oplus \left\langle \frac{\partial}{\partial p_i} \right\rangle$$

Universal Prolongation

For every nilpotent Lie algebra $\mathfrak{m}$ there exists unique universal prolongation $\mathfrak{g}(\mathfrak{m}) = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$, that:

1. $\mathfrak{g}_i = \mathfrak{m}_i$, for all $i < 0$
2. From $[X, \mathfrak{g}_{-1}] = 0$, $X \in \mathfrak{g}_i, i \geq 0$ it is follows that $X = 0$.

Universal prolongation $\mathfrak{g}(\mathfrak{m})$ for the equation (1) is called a symbol of the differential equation (1).

Fact (Tanaka,Marimoto)

For every filtered manifold we can build a normal Cartan connection of type $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{h} = \oplus_{i\geq 0}\mathfrak{g}_i$.

Fact (Tanaka,Marimoto)

All invariants of the filtered manifold arise from the curvature tensor of the normal Cartan connection. Fundamental system of invariants described by positive part of cohomology space $H^2(\mathfrak{g}_-, \mathfrak{g})$.

Symbol Lie algebra

Universal prolongation $\mathfrak{g}$ of Lie algebra $\mathfrak{m}$:

$$\mathfrak{g} = (\mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{gl}_m(\mathbb{R})) \ltimes (V_2 \otimes \mathbb{R}^m).$$

We fix the following basis in the Lie algebra $\mathfrak{sl}_2$

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We fix basis e_0, e_1, e_2 of module V_2 with relations $xe_i = e_{i-1}$.

$\mathfrak{g}_1 = \langle y \rangle$, $\mathfrak{g}_0 = \langle \mathfrak{gl}_m \rangle$,

$\mathfrak{g}_{-1} = \langle x \rangle + \langle e_2 \otimes \mathbb{R}^m \rangle$, $\mathfrak{g}_{-2} = \langle e_1 \otimes \mathbb{R}^m \rangle$, $\mathfrak{g}_{-3} = \langle e_0 \otimes \mathbb{R}^m \rangle$.

"Parabolic" subalgebra:

$$\mathfrak{h} = \mathfrak{g}_1 + \mathfrak{g}_0$$

Cartan Connection

We call co-frame $\omega_x, \omega^i_{-1}, \omega^i_{-2}, \omega^i_{-3}$ associated with equation (1) if

$$C^{-2} = < \omega^i_{-3} >^t$$
$$C^{-1} = < \omega^i_{-3}, \omega^i_{-2} >^t$$
$$E = < \omega^i_{-3}, \omega^i_{-2}, \omega^i_{-1} >^t$$
$$V = < \omega^i_{-3}, \omega^i_{-2}, \omega^i_x >^t$$

Let $\pi : P \to \mathcal{E}$ be principle H-bundle. We say that Cartan connection ω

$$\omega = \omega^i_{-3} v_0 \otimes e_i + \omega^i_{-2} v_1 \otimes e_i + \omega^i_{-1} v_2 \otimes e_i + \omega^x x + \omega^h h + \omega^i_j e^j_i + \omega^h h.$$

on a principal H-bundle is associated to equation (1), if for any local section s of π the set $\{s^*\omega_x, s^*\omega^i_{-1}, s^*\omega^i_{-2}, s^*\omega^i_{-3}\}$ is an adapted co-frame on $\mathcal{E}$.

Normal Form

Let the curvature have a form

$$\Omega = \Omega^i_{-3} v_0 \otimes e_i + \Omega^i_{-2} v_1 \otimes e_i + \Omega^i_{-2} v_2 \otimes e_i + \Omega^x x + \Omega^h h + \Omega^i_j e^j_i + \Omega^y y$$

Let's $\Omega^j_k[\omega_1, \omega_2]$ be the coefficients of the structure function of the curvature Ω.

Theorem

There exist the unique normal Cartan connection associated with equation (1) with the following conditions on curvature:

- *all components of degree 0 and 1 is equal to 0;*
- *in degree 2 we have* $\Omega_h[\omega_x \wedge \omega^i_{-1}] = 0$, $\Omega^i_j[\omega_x \wedge \omega^i_{-1}] = 0$, $\Omega_x[\omega_x \wedge \omega^i_{-2}] = 0$, $\Omega^i_{-1}[\omega_x \wedge \omega^i_{-2}] = 0$;
- *in degree 3 we have* $\Omega_y[\omega_x \wedge \omega^i_{-1}] = 0$, $\Omega_h[\omega_x \wedge \omega^i_{-2}] = 0$, $\Omega^i_j[\omega_x \wedge \omega^i_{-2}] = 0$
- *in degree 4 it is* $\Omega_y[\omega_x \wedge \omega^i_{-2}] = 0$

Serre spectral sequence

We use Serre spectral sequence to compute cohomology space.

Theorem

The space of cohomology classes $H^2(\mathfrak{g}_-, \mathfrak{g})$ is direct sum of two spaces $E_2^{1,1}$ and $E_2^{0,2}$, where $E_2^{1,1}$ and $E_2^{0,2}$ has the following form

$$E_2^{1,1} = H^1(\mathbb{R}x, H^1(V, \mathfrak{g})), \quad (5)$$

$$E_2^{0,2} = H^0(\mathbb{R}x, H^2(V, \mathfrak{g})). \quad (6)$$

Description of $E_2^{1,1}$ and $E_2^{0,2}$.

Let V_m be irreducible $\mathfrak{sl}_2$-module and v_0 and v_m such, that $x.v_0 = 0$ and $y.v_m = 0$. Then

$$H^0(\mathbb{R}x, V_m) = \mathbb{R}v_0$$

$$H^1(\mathbb{R}x, V_m) = \mathbb{R}x^* \otimes v_m.$$

Structure of the space $E_2^{1,1}$

Theorem

The space $E_2^{1,1}$ has the following form

$$x^* \otimes (\mathbb{R}y^2 \otimes \mathfrak{gl}(W) + \mathbb{R}y \otimes \mathfrak{sl}(W)). \tag{7}$$

Elements $\varphi \in E_2^{1,1}$ which have form $\varphi\colon \mathbb{R}x \to \mathbb{R}y \otimes \mathfrak{sl}(W)$ have degree 2 and elements which have form $\varphi\colon \mathbb{R}x \to \mathbb{R}y^2 \otimes \mathfrak{gl}(W)$ have degree 3

- The space $E_2^{1,1}$ describes so called Wilczynski invariants.
- The space $E_2^{1,1}$ corresponds to torsion.

Structure of the space $E_2^{0,2}$

Theorem

The space $E_2^{0,2}$ has the following parts in direct sum decomposition:

Space	Degree
$V_6 \otimes \wedge^2(W^*) \otimes W$	-1
$V_4 \otimes S_0^2(W^*) \otimes W$	0
$V_4 \otimes \wedge^2(W^*) \otimes W$	0
$V_2 \otimes \wedge_0^2 W^* \otimes W$	1
$V_0 \otimes S_0^2(W^*) \otimes W$	2
$V_0 \otimes S^2(W^*)$	4
$V_2,\ m = 2$	3

where V_m is dimension $m+1$ $\mathfrak{sl}_2$-module and $S_0^2(W^*) \otimes W$ and $\wedge_0^2(W^*) \otimes W$ is traceless part of corresponding spaces

Explicit formula

We have the following correspondence between cohomological classes $H^2(\wedge^2\mathfrak{g}_-,\mathfrak{g})$ and fundamental invariants:

Degree	Space	Invariant
1	$v_2^0 \otimes \wedge^2 W^* \otimes W / V_2 \otimes W^*$	$\equiv 0$
2	$x^* \otimes \mathbb{R}y \otimes \mathfrak{sl}(W)$	W_2
2	$v_0^0 \otimes S^2(W^*) \otimes W$	I_2
3	$x^* \otimes \mathbb{R}y^2 \otimes \mathfrak{gl}(W)$	W_3
4	$v_0^0 \otimes S^2(W^*)$	I_4
3	v_2^0 if $m = 2$	$\equiv 0$

Explicit formula

Theorem

There are 4 fundamental invariants:

$$W_2 = \mathrm{tr}_0 \left(\frac{\partial f^i}{\partial p^j} - \frac{\mathrm{d}}{\mathrm{dx}} \frac{\partial f^i}{\partial q^j} + \frac{1}{3} \frac{\partial f^i}{\partial q^k} \frac{\partial f^k}{\partial q^j} \right)$$

$$I_2 = \mathrm{tr}_0 \left(\frac{\partial^2 f^i}{\partial q^j \partial q^k} \right)$$

$$W_3 = \frac{\partial f^i}{\partial y^j} - \frac{1}{3} \frac{\mathrm{d}^2}{\mathrm{dx}^2} \frac{\partial f^i}{\partial q^j} - \frac{1}{27} (\frac{\partial f^i}{\partial q^k})^3 + \frac{2}{9} \frac{\partial f^i}{\partial q^k} \frac{\mathrm{d}}{\mathrm{dx}} \frac{\partial f^k}{\partial q^j} + \frac{1}{9} \frac{\mathrm{d}}{\mathrm{dx}} \frac{\partial f^i}{\partial q^k} \frac{\partial f^k}{\partial q^j}$$

$$I_4 = \frac{\partial}{\partial q_k} \frac{\mathrm{d}}{\mathrm{dx}} H_j + \frac{\partial}{\partial q^k} (H_l \frac{\partial f^k}{\partial q^l}) + \frac{\partial H_k}{\partial p_j}$$

where $H_j = tr \left(\frac{\partial^2 f^i}{\partial q^j \partial q^k} \right)$

Conformal geodesics

An arbitrary conformal geometry is defined by equivalence class of Riemannian metrics.

Definition

Conformal geodesic on a conformal manifold M is a curve on M, which development in the flat space is a circle.

Example

$$\dddot{y}_i = 3\ddot{y}_i \frac{\sum_{j=1}^{m} \dot{y}_j \ddot{y}_j}{1 + \sum_{j=1}^{m} \dot{y}_j^2}, i = 1, \ldots, m.$$

Theorem (Yano)

In order that an infinitesimal transformation of the manifold M carry every conformal geodesic into conformal geodesic, it is necessary and sufficient that the transformation be a conformal motion.

Correspondence Space Construction

Penrose transform

Consider a semisimple Lie group G with two parabolic subgroups P_1 and P_2. Assume, that $P_1 \cap P_2$ is also parabolic.
Then a natural double fibration from $G/P_1 \cap P_2$ to G/P_1 and G/P_2 defines a correspondence between G/P_1 and G/P_2.

Correspondence space

let G be a semisimple Lie group with two parabolic subgroups $Q \subset P \subset G$. Consider a parabolic Cartan geometry $(\mathcal{G} \to N, \omega)$ of type (G, P), where $\mathcal{G}$ is principal P-bundle.

Definition

The *correspondence space* of a parabolic geometry $(\mathcal{G} \to N, \omega)$ is the orbit space $\mathcal{C}N = \mathcal{G}/Q$.

Cartan geometry $(\mathcal{G} \to \mathcal{C}N, \omega)$ of the type (G, Q) is naturally defined on the correspondence space $\mathcal{C}N$.

Correspondence Space Construction

Consider a flat conformal geometry of the dimension $m+1$. Define a quadratic form $q_L(x)$ on Lorenzian space $L = R^{m+3}$ by the formula

$$q_L(x) = -2x_0 x_{m+2} + x_1^2 + \dots x_{m+1}^2. \qquad (8)$$

The vector V is called light-like if $q_L(V) = 0$. The space M of light-like points in PL is called Mobius space. This space is homogenious:

$$M = SO_{m+2,1}/P, \qquad (9)$$

where P is a stabilizer of a light-like vector in PL. Lie algebras $\mathfrak{so}_{m+2,1}$ and $\mathfrak{p}$ of groups $SO_{m+2,1}$ and P respectively have the following forms:

$$\mathfrak{g} = \begin{pmatrix} z & q & 0 \\ p & r & q^t \\ 0 & p^t & -z \end{pmatrix}; \qquad \mathfrak{p} = \begin{pmatrix} z & q & 0 \\ 0 & r & q^t \\ 0 & 0 & -z \end{pmatrix} \qquad (10)$$

Correspondence Space Construction

We use the following notations:

$$\mathfrak{g} = \begin{pmatrix} \tilde{h} & \tilde{y} & q & 0 \\ \tilde{x} & 0 & -s^t & \tilde{y} \\ p & s & r & q^t \\ 0 & \tilde{x} & p^t & -\tilde{h} \end{pmatrix}; \qquad \mathfrak{p} = \begin{pmatrix} \tilde{h} & \tilde{y} & q & 0 \\ 0 & 0 & -s^t & \tilde{y} \\ 0 & s & r & q^t \\ 0 & 0 & 0 & -\tilde{h} \end{pmatrix} \quad (11)$$

Consider conformal Cartan connection ω on a principal P-bundle $\pi : \mathcal{P} \to M$.

We define new bundle $\tilde{\pi} : \mathcal{P} \to \tilde{M}$ with the same total space $\mathcal{P}$ and new base $\tilde{M} = \mathcal{P} \times_{P_2} P_2$, where P_2 is the Lie group with Lie algebra $\mathfrak{p}_2$

$$\mathfrak{p}_2 = \begin{pmatrix} \tilde{h} & \tilde{y} & 0 & 0 \\ 0 & 0 & 0 & \tilde{y} \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & -\tilde{h} \end{pmatrix}$$

Extension Functor

Let $(\mathfrak{g}, \mathfrak{h})$ be a Cartan geometry type of 3rd order ODEs system. Define a map $\alpha : \mathfrak{so}_{m+2,1} \to \mathfrak{g}$ which sends $\mathfrak{p}_2$ to the $\mathfrak{h}$ with property:

$$\alpha([g_1, p_1]) = [\alpha(g_1), \alpha(p_1)], \quad g_1 \in \mathfrak{so}_{m+2,1}, p_1 \in \mathfrak{p}_2.$$

Explicitly it has the form:

$$\alpha\left(\begin{pmatrix} \tilde{h} & \tilde{y} & q & 0 \\ \tilde{x} & 0 & -s^t & \tilde{y} \\ p & s & r & q^t \\ 0 & \tilde{x} & p^t & -\tilde{h} \end{pmatrix}\right) = \begin{pmatrix} -\frac{1}{2}\tilde{h} & \tilde{x} \\ \frac{1}{2}\tilde{y} & \frac{1}{2}\tilde{h} \end{pmatrix} + (r) + \left(v_0 \otimes p - v_1 \otimes s + v_2 \otimes q\right), \quad (12)$$

where $r \in \mathfrak{gl}_m$ and $v_0 \otimes p$, $v_1 \otimes s$, $v_2 \otimes q$ is elements of $V_2 \otimes W$.

Extension Functor

- Starting with the principal P_2-bundle $\mathcal{P}$ define new principal H-bundle $\tilde{\mathcal{P}}$ by the formula:

$$\tilde{\mathcal{P}} = \mathcal{P} \times_{P_2} H, \tag{13}$$

where inclusion of group P_2 to H is defined by α.

- We define Cartan connection $\tilde{\omega} : \tilde{P} \to \tilde{\mathfrak{g}}$ as H-eqivariant prolongation of $\tilde{\omega} = \alpha(\omega)$.
- The curvature of the Cartan connection $\tilde{\omega}$ is

$$\tilde{\Omega} = \Omega + R, \tag{14}$$

where $R(x, y) = \alpha([x, y]) - [\alpha(x), \alpha(y)]$.

Conditions on conformal equations

The following commutative diagram describes a geometric picture we have:

$$\begin{array}{ccccc} \mathcal{P} & = & \mathcal{P} & \longrightarrow & \tilde{\mathcal{P}} \\ \downarrow P & & \downarrow P_2 & & \downarrow H \\ M & \longleftarrow & \tilde{M} & = & \tilde{M} \end{array}$$

Theorem

Map defined above sends conformal geodesics of the manifold M to the solution of the associated equation of the manifold $\tilde{M}$.

Theorem

Map defined above sends normal Cartan conformal connection to the characteristic connection of the 3rd order ODEs system.

Corollary

Every conformal geometry is locally defined by conformal geodesics.

Conditions on conformal equations

Theorem

On the inclusion defined above a second order Wilczynski invariant is expresed in terms of the Weil tensor; a third order Wilczynski invariant is expresed in terms of the Cotton-York tensor.

Proposition

The invariant I_2 is equal to zero for conformal equations. Invariant I_4 must be non-degenerate function from $\mathrm{Hom}(\tilde{\mathcal{P}}, S^2(R^{m*}))$

Proposition

The tensor R_α goes exactly to the I_4 invariant. Moreover, tensor R_α corresponds to the identity form $E \in S^2(R^{m*})$.

The first reduction

- Let $\tilde{\mathcal{P}}_1$ be subbundle of the bundle $\tilde{\mathcal{P}}$ on which I_4 equals to E.
- Let $\tilde{\omega}_1$ be the induced Cartan connection on $\tilde{\mathcal{P}}_1$.
- The connection $\tilde{\omega}_1$ must take values in the $\mathfrak{so}_{m+2,1} \subset \tilde{\mathfrak{g}}$ in order to determine conformal geometry.

Algebra $\mathfrak{g}$ splits as $\mathfrak{h}$-module:

$$\mathfrak{gl}_m \oplus \mathfrak{sl}_2 \oplus V_2 \otimes W$$

This induces the split of the universal derivative

$$D = D_{\mathfrak{gl}_m} + D_{\mathfrak{sl}_2} + D_{V_2 \otimes W}$$

Theorem

The connection $\tilde{\omega}_1$ determines the connection on $T\tilde{\mathcal{P}}_1$ with values in $\mathfrak{so}_{m+2,1}$ iff $D_{\mathfrak{sl}_2}(I_4) = 0$ and $D_{V_2 \otimes W}(I_4) = 0$.

The second reduction

- The bundle $\tilde{\mathcal{P}}_1$ is the bundle over $\tilde{M}$.
- We want to determine when it can be seen as bundle over M

Let X_i be the vector fields which represent $\mathfrak{h}/\mathfrak{p}_2$.
Connection $\tilde{\omega}_1$ is a connection on the bundle $\mathcal{P} \to M$ iff

$$i_{X_i}\tilde{\Omega}_1 = 0.$$

Theorem

The 3rd order ODEs system determines conformal geodesics of some conformal geometry iff the following conditions are satisfied:

1. *Invariant I_2 equal to zero;*
2. *Invariant I_4 has the maximal rank and $D_{\mathfrak{sl}_2}(I_4) = 0$, $D_{V_2 \otimes W}(I_4) = 0$;*
3. $i_{e_1 \otimes W}\tilde{\Omega} = 0$ *and* $i_{e_2 \otimes W}\tilde{\Omega} = 0$.