Higher order higher spin conformally invariant operators on the sphere

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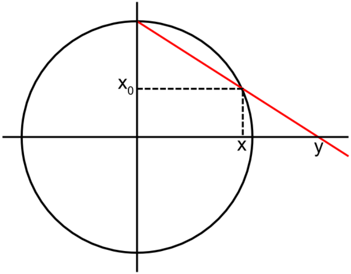
Outline

- Conformally invariant powers of the Laplace operator
- Higher spin representations and operators
- Existence of conformally invariant higher spin Dirac operators
- Construction by spectrum generating method and examples

Stereographic projection

[Graham, 2007] showed a nice way how to calculate conformally invariant powers of the Laplace operator on the sphere using the stereographic projection:



$$\Phi : S^n \setminus \{(1,0)\} \to \mathbb{R}^n$$

$$y := \Phi((x_0, x)) = \frac{x}{1 - x_0}$$

$$\Phi^* \left(\frac{2}{(1 + |y|^2)} \right) = 1 - x_0 =: \Omega_S$$

$$\Phi^* g_E = \frac{1}{(1 - x_0)^2} g_S$$

Here g_E is the standard Euclidean metric on $\mathbb{R}^n$ and g_S the standard metric on the sphere S^n. Hence Φ is conformal. We denote by Δ the Laplace operator on $\mathbb{R}^n$ and by Δ_S the Laplace operator on S^n.

Conformally invariant operators

The **Yamabe operator** $Y := \Delta_S - \frac{n(n-2)}{4}$ is conformally invariant, thus

$$Y \Omega_S^{1-\frac{n}{2}} \Phi^* = \Omega_S^{-1-\frac{n}{2}} \Phi^* \Delta$$

The powers of Δ are conformally invariant too, for any $C : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $C^* g_E = \Omega^2 g_E$ it holds

$$\Delta^k \Omega^{k-\frac{n}{2}} C^* = \Omega^{-k-\frac{n}{2}} C^* \Delta^k$$

Conformal transformations of S^n are conjugations of C by Φ, hence we only seek a formula of the type

$$Y^{(k)} \Omega_S^{k-\frac{n}{2}} \Phi^* = \Omega_S^{-k-\frac{n}{2}} \Phi^* \Delta^k$$

and the conformal invariance of $Y^{(k)}$ follows.

Conformally invariant "powers" of the Laplacian

The right expression is

$$Y^{(k)} = \prod_{j=1}^{k} (Y + j(j-1)) = \prod_{j=1}^{k} \left(\Delta_S - \left(\frac{n}{2} + j - 1 \right) \left(\frac{n}{2} - j \right) \right)$$

and the proof follows by induction from the commutation relation of Δ and $(\Phi^* \Omega_S)^w \equiv (2/(1+|y|^2))^w$, $w \in \mathbb{R}$. Note that $Y^{(k)} = (\Delta_S)^k + LOTS$. On a general manifold, $LOTS$ depend on curvature and $Y^{(k)}$'s are the **GJMS operators**, constructed by the ambient space method. Also note **the product structure** (only on S^n) of $Y^{(k)}$, which we'll see in further examples too.

Higher spin representations

Spinor representation (assume odd n for simplicity) is an irreducible $\mathrm{Spin}(n)$ representation with highest weight $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$. More complicated half-integral representations occur in **physics** (Rarita-Schwinger, Penrose,...), **geometry** (Branson, Somberg,...) and **Clifford analysis** (Delanghe, Souček, Bureš, Ryan, Sommen, Brackx, Van Lancker, Eelbode, Van de Voorde, Raeymaekers,...)

The simplest example is

$$(k)' := \left(k + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$$

Stein-Weiss gradients

For λ a highest weight of an irreducible Spin(n)-module, denote the associated vector bundle $\mathbb{V}(\lambda)$. A **Stein-Weiss gradient** is a composition

$$G_u := G_{\lambda\sigma_u} : \Gamma(\mathbb{V}(\lambda)) \xrightarrow{\nabla} \Gamma(T^*M \otimes \mathbb{V}(\lambda)) \xrightarrow{\text{proj}} \Gamma(\mathbb{V}(\sigma_u))$$

of the Levi-Civita connection and a projection onto a direct summand. It is the unique (up to a multiple) Spin(n)-invariant differential operator from $\mathbb{V}(\lambda)$ to $\mathbb{V}(\sigma_u)$.

Examples:

- $\lambda = \sigma_u = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)$ Dirac operator
- $\lambda = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)$,$\sigma_u = \left(\frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right)$ twistor operator
- $\lambda = \left(\frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right)$,$\sigma_u = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)$ dual twistor operator
- $\lambda = \sigma_u = \left(\frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right)$ Rarita-Schwinger operator

Higher spin first order operators

We are interested in Stein-Weiss gradients

$$D_k : \Gamma(\mathbb{V}((k)')) \to \Gamma(\mathbb{V}((k)'))$$

(**Higher spin Dirac operator**),

$$T_{k:k-1} : \Gamma(\mathbb{V}((k)')) \to \Gamma(\mathbb{V}((k-1)'))$$

(**Higher spin twistor operator**) and

$$T^*_{k:k-1} : \Gamma(\mathbb{V}((k-1)')) \to \Gamma(\mathbb{V}((k)')),$$

the formal adjoint of $T_{k:k-1}$.

Every Stein-Weiss gradient (and hence every higher spin first order operator) is conformally invariant with a suitable conformal weight (Fegan, 1976).

Conformal invariance in higher order

Compositions of conformally invariant operators are not conformally invariant because conformal weights do not match. Classification of conformally invariant operators was done by [Slovák, 1993] via their identification with homomorphisms of generalized Verma modules. For the case of higher spin Dirac operators the following can be deduced:

Theorem ([Š., 2012])

Let $n = 2m + 1$ be odd, $\lambda = (k_1, k_2, \ldots, k_{m-1}, 0) + \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ a dominant weight. Then there is a conformally invariant p-th order operator $D : \Gamma(\mathbb{V}(\lambda)) \to \Gamma(\mathbb{V}(\lambda))$ ***if and only if*** *p* ***is odd****.*

This covers almost all higher spin representations.

Conformally invariant powers of Dirac

For the Dirac case ($\lambda = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)$) the operators are $\forall N \in \mathbb{Z}_0^+$

$$D_0\left((D_0)^2 - 1\right)\left((D_0)^2 - 4\right)\dots\left((D_0)^2 - N^2\right),$$

where D_0 is the Dirac operator on S^n. This was found by Liu and Ryan (2002) by the stereographic method, by Eelbode and Souček (2010) by the ambient method and by Branson and Ørsted (2006) by the spectrum generating method. Note again the product structure.

Moreover, Holland and Sparling (2001) proved existence of the curved versions by the ambient space method.

The spectrum generating method

Branson, Ólaffson and Oersted (1996) introduced a method of **calculation of spectra** of conformally invariant operators.

The conformal sphere is a parabolic geometry G/P, but as a Riemannian manifold it is K/M, where $K = \mathrm{Spin}(n+1)$, $M = \mathrm{Spin}(n)$ are subgroups of $G = \mathrm{Spin}_0(n+1,1)$, $P = MAN$, $A = \mathbb{R}^+$, $N = \mathbb{R}^{n+1}$. The space of K-finite sections of $\Gamma(\mathbb{V}(\lambda)^r)$ is interpreted as a $(\mathfrak{g}, K)$-module and it decomposes into irreducible K-modules (**K-types**) according to a branching rule. The proper conformal Killing fields in $\mathfrak{g}$ then map between different K-types.

An operator D is conformally invariant iff it intertwines the $(\mathfrak{g}, K)$-action. The K-invariance means that it is **constant** on each K-type, the $\mathfrak{g}$-invariance determines the **ratios of eigenvalues** on pairs of K-types, hence the spectrum up to an overall multiple.

Spectrum generating applied to Dirac

For $\lambda = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$, the K-type decomposition is

$$\Gamma(\mathbb{V}(\lambda)) = \bigoplus_{\epsilon=\pm 1} \bigoplus_{j=0}^{\infty} \mathcal{V}_\epsilon(j),$$

where $\mathcal{V}_\epsilon(j)$ is an irreducible $\mathrm{Spin}(n+1)$-module with highest weight $\left(j + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{\epsilon}{2}\right)$

A conformally invariant operator of **order 1** has to have eigenvalue $\epsilon \frac{n+2j}{n}$ on $\mathcal{V}_\epsilon(j)$ and up to a multiple it has to be the Dirac operator D_0.

Similarly, the spectral method calculates eigenvalues of a conformally invariant operator of **order 2N+1**, expresses them in terms of eigenvalues of Dirac and obtain the formula given earlier

$$R_{0,2N+1} = D_0 \left((D_0)^2 - 1\right)\left((D_0)^2 - 4\right) \ldots \left((D_0)^2 - N^2\right)$$

Conformal invariants on spinor-forms

Hong (2011) found the expression for the conformally invariant operators for $\lambda = \left(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$. The K-type decomposition is now

$$\Gamma(\mathbb{V}(\lambda)) = \bigoplus_{\epsilon=\pm 1} \bigoplus_{j=1}^{\infty} \mathcal{V}_\epsilon(j,1) \oplus \mathcal{V}_\epsilon(j,0)$$

for $\mathcal{V}_\epsilon(j,q) = \left(j+\frac{1}{2}, q+\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{\epsilon}{2}\right)$ and the operator of order $2N+1$ has the form

$$R_{1,2N+1} = D_1 \prod_{l=1}^{N} \left(D_1^2 - l^2 \,\mathrm{Id} - \frac{16 l^2}{n(n^2-4l^2)} T_{1:0}^* T_{1:0} \right)$$

The highest order term in each bracket is a linear combination of D_1^2 and $T_{1:0}^* T_{1:0}$ that span the space of K-invariants.

Conformal invariants for $\frac{5}{2}$ spin: first half

Similar approach leads to expressions for conformally invariant operators for $\lambda = \left(\frac{5}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right)$. The eigenvalues for an operator of **order 4N+1** are calculated with respect to the K-type decomposition

$$\Gamma(\mathbb{V}(\lambda)) = \bigoplus_{\epsilon=\pm 1} \bigoplus_{j=2}^{\infty} \mathcal{V}_\epsilon(j,2) \oplus \mathcal{V}_\epsilon(j,1) \oplus \mathcal{V}_\epsilon(j,0)$$

The resulting operator is expected in the form

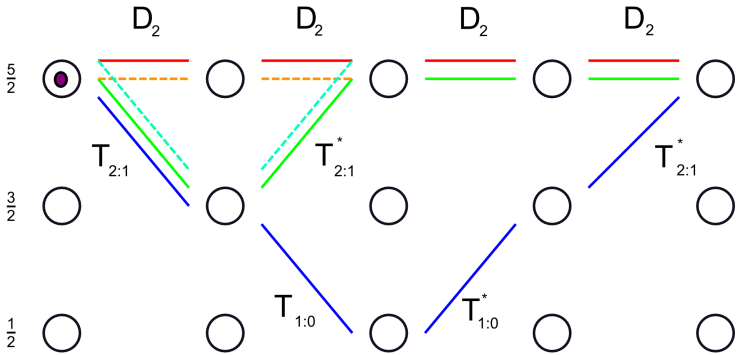
$$R_{2,4N+1} := D_2 A_{2,1} A_{2,2} \dots A_{2,N},$$

with the **4th order** inhomogeneous operators

$$A_{2,l} := \left((D_2)^2 - (2l)^2 \operatorname{Id}\right)\left((D_2)^2 - (2l-1)^2 \operatorname{Id}\right) + \\ + a_l T^*_{2:1} T^*_{1:0} T_{1:0} T_{2:1} + b_l D_2^2 T^*_{2:1} T_{2:1} + c_l T^*_{2:1} T_{2:1}$$

as building blocks, for some values of $a_l, b_l, c_l \in \mathbb{R}$.

The ingredients



Actual values of coefficients

This leads to N systems of 6 linear equations for 3 unknowns a_l, b_l, c_l. In accordance with general theory, there is a (unique) solution for every $l \in \{1, \ldots, N\}$, of the form

$$a_l = \frac{256(n-1)(2l)^2(2l-1)^2}{(n+2)(n+4l)(n+4l+2)(n-4l+4)(n-4l+2)}$$

$$b_l = -\frac{16(n^2(8l^2-4l+1)+n(16l^2-8l+2)+16l^2-8l)}{n(n+2)(n+4l+2)(n-4l+4)}$$

$$c_l = \frac{32n(2l)^2(2l-1)^2}{(n+2)(n-4l+4)(n+4l+2)}$$

Conformal invariants for $\frac{5}{2}$ spin: second half

In a similar way an operator

$$R_{2,4N+3} := R_{2,3}\tilde{A}_{2,1}\tilde{A}_{2,2}\dots\tilde{A}_{2,N},$$

can be constructed from the **3rd order** operator

$$R_{2,3} := D_2\left((D_2)^2 - \mathrm{Id} - \frac{16}{(n+2)(n+4)}T^*_{2:1}T_{2:1}\right)$$

and the **4th order** operators

$$\tilde{A}_{2,l} := \left((D_2)^2 - (2l)^2\,\mathrm{Id}\right)\left((D_2)^2 - (2l+1)^2\,\mathrm{Id}\right) + \\ + \tilde{a}_l T^*_{2:1}T^*_{1:0}T_{1:0}T_{2:1} + \tilde{b}_l D_2^2 T^*_{2:1}T_{2:1} + \tilde{c}_l T^*_{2:1}T_{2:1}$$

for appropriate values of $\tilde{a}_l$, $\tilde{b}_l$, $\tilde{c}_l$.

Summary and outlook

- The goal was to generalize "powers" of Laplacian and Dirac to higher spin representation.
- We restricted to the sphere, curved versions seem very complicated.
- Moreover, on the sphere we have the spectrum generating method, which gives the operators easily.
- The operators exist in every odd order and come as products of low order (not conformally invariant) operators.
- Raising spin leads to more K-types and hence unknowns and conditions, but the general pattern seems to be preserved. Namely, we can calculate all orders at once with a system of linear equations with the size depending on the highest weight of the representation.

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