Transfinite powers in rings of differential polynomials

joint work with G. Puninski

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Iterated power intersections of an ideal

Definition

Let R be a ring and let I be an ideal. By transfinite induction we define the following sequence of ideals:

1. $I(0) = I$
2. $I(\alpha + 1) = \cap_{n\in\mathbb{N}} I(\alpha)^n$
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Corollary

1. *Let R be a commutative noetherian domain, I a proper ideal. Then $I(1) = 0$.*
2. *Let R be a commutative noetherian ring and $I = I^2$. Then $I = eR$ for some idempotent $e \in R$.*

A similar concept has been studied by P. Smith who defined a sequence $\kappa_n(I)$, $n \in \mathbb{N}_0$, where $\kappa_0(I) = I$ and $\kappa_{n+1}(I)$ is the greatest ideal such that $\kappa_{n+1}(I)\kappa_n(I) = \kappa_{n+1}(I)$.

Projective modules over some noetherian rings

When study countably but not finitely generated modules over a (left and right) noetherian rings one is interested if the ring satisfies the following condition (*): There is no infinite strictly descending chain of ideals $I_0 \supsetneq I_1 \supsetneq \cdots$ such that $I_{n+1} I_n = I_{n+1}$. When this condition holds, one can classify countably generated projective modules by pairs (I, P), where I is an idempotent ideal of R and P is a finitely generated projective module over R/I.

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Remark

1. *Suppose we have such a sequence $I_0, I_1, \ldots$ If we put $I = I_0$, then an easy induction shows that $I_n \subseteq I(n)$.*
2. *Suppose that for every maximal ideal I there exists n such that $I(n) = 0$. Then the condition (*) holds and the only idempotent ideals of R are 0 and R. Then every countably generated projective R-module is either finitely generated or free.*

The following results can be seen as a consequence of 2.

Theorem

1. *(Bass) If R is a simple noetherian ring, then every countably but not finitely generated projective module is free.*
2. *(Bass) If R is a connected commutative noetherian ring, then every countably but not finitely generated projective module is free.*
3. *(Swan) If G is a finite solvable group, then every countably but not finitely generated projective $\mathbb{Z}G$-module is free.*
4. *(P.) If L is a finite dimensional solvable Lie algebra over a field of characteristic 0. Then every countably but not finitely generated projective $U(L)$-module is free.*

Rings of differential polynomials

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Let S be a ring. A derivation on S is an additive map $\delta\colon S \to S$ satisfying the Leibniz rule $\delta(ab) = \delta(a)b + a\delta(b)$. If there exists $d \in S$ such that $\delta(x) = dx - xd$ for every $x \in S$, then δ is an inner derivation on S.

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Definition

Let S be a ring and let $\delta\colon S \to S$ be a derivation on S. A ring of differential polynomials $S[y, \delta]$ is a free right S-module with basis $1, y, y^2, \ldots$ and a multiplication given by the rule $sy = ys - \delta(s), s \in S$.

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Every element of $S[y, \delta]$ has a unique expression as
$y^n s_n + y^{n-1} s_{n-1} + \cdots + s_0$ and also as $s'_n y^n + s'_{n-1} y^{n-1} + \cdots + s'_0$, where $s_n = s'_n$.

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Example

Let k be a field, $S = k[x]$ and $\delta\colon S \to S$ the standard derivation. $S[y, \delta] = A_1(k)$ is then the first Weyl algebra over k.

Rings of iterated differential polynomials

Iteration of this construction gives rings of iterated differential polynomials $S[y_1, \delta_1, y_2, \delta_2, \ldots, y_n, \delta_n]$ (δ_i is a derivation on $S[y_1, \delta_1, \ldots, y_{i-1}, \delta_{i-1}]$)

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Example

Let $S = k[x_1, \ldots, x_n]$. For $i = 0, \ldots, n$ define R_i: $R_0 = S$ and $R_i = R_{i-1}[S, y_i, \partial/\partial_i]$. Then $R_n = A_n(k)$ is the n-th Weyl algebra over k. If k is a field of characteristic zero then $A_n(k)$ is a simple noetherian domain.

Some standard properties

Lemma

If T is an Ore set in S consisting of nonzerodivisors, then any derivation on S can be uniquely extended to a derivation on ST^{-1} by $t^{-1} \mapsto -t^{-1}\delta(t)t^{-1}, t \in T$.

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If S is a skew field and δ is a derivation on S, then $S[y, \delta]$ is a left and right principal ideal domain. In particular $I(1) = 0$ for every proper ideal $I \subseteq S[y, \delta]$.

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Theorem

Let S be a commutative noetherian domain that is a $\mathbb{Q}$-algebra. Let $R = S[y_1, \delta_1, \ldots, y_n, \delta_n]$ be a ring of iterated differential polynomials. Then every prime ideal of R is completely prime.

The result

Theorem

Let S be a commutative noetherian domain that is a $\mathbb{Q}$-algebra. Let I be a proper ideal in the ring of iterated differential polynomials $R = S[y_1, \delta_1, \dots, y_n, \delta_n]$. Then $I(n+1) = 0$.

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Corollary

If R is as above then every countably but not finitely generated projective module is free.

A prominent example

Definition

Let L be a Lie algebra over a field k, let B be a basis of L. The universal enveloping algebra of L is the algebra $U(L) = k\langle B\rangle/(b_i b_j - b_j b_i = [b_i, b_j] \mid b_i, b_j \in B)$.

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Theorem

Let L be a finite dimensional solvable Lie algebra over an algebraically closed field k of characteristic zero. Then there exists a basis $\{b_1, \dots, b_n\}$ *such that for every* $i = 1, \dots, n$ *the space* $kb_1 + \cdots kb_i$ *is an ideal of L. In particular,* $U(L)$ *can be seen as a ring of iterated differential operators.*

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Theorem

Let L be a finite dimensional solvable Lie algebra over a field of characteristic zero. Then $I(2) = 0$ for every proper ideal I of $U(L)$.

A strange connection

Proposition

Let R be a noetherian k-algebra. Let V be a finite dimensional simple module. Then $\mathrm{Ann}_R V$ is idempotent, if $\mathrm{Ext}^1_R(V, V) = 0$. So if S is a noetherian $\mathbb{Q}$-algebra, $R = S[y_1, \delta_1, \ldots, y_n, \delta_n]$, then all finite dimensional simple R-modules have nontrivial self-extensions.