# A MODEL STRUCTURE ON THE CATEGORY OF TOPOLOGICAL CATEGROIES

AMRANI ILIAS

Abstract. We construct a cofibrantly generated Quillen model structure on the category of small topological categories $\mathbf{Cat_{Top}}$. It is Quillen equivalent to the Joyal model structure of $(\infty, 1)$-categories and the Bergner model structure on the category of simplicial categories $\mathbf{Cat_{sSet}}$. We also give a new complete description for the mapping space of the model category $\mathbf{Cat_{Top}}$.

## Introduction

In t section 1, we describe in details the Quillen model structure on the category of small topological categories $\mathbf{Cat_{Top}}$ [1] and we give the strategy of proof without technical details, which will be carefully carried out in the next sections. As the experience shows, it is better to have equivalent model structure and to be able to jump from one to another. For instance, the Joyal model structure on simplicial sets for quasi-categories is very convenient, it is a symmetric monoidal combinatorial model category and the pushouts are easy to compute. But we loose our intuition if we want to apply some categorical construction such as limits and colimits for a given quasi-category. An other interesting Quillen equivalent model structure was constructed by Bergner [2] for the category of small simplicial categories. This category is well behaved, but it is not a symmetric monoidal model category and the objects are not always fibrant. However its homotopy category is a symmetric monoidal closed category [11]. That is why we propose an other alternative which is the category of topological small categories. It is the worse category from the categorical point of view because it is not combinatorial anymore but *all objects are fibrant*. This is a key point to construct a model structure on the category of topological categories with some extra algebraic structure (for example symmetric monoidal enriched categories).
One of the major difficulty to prove the main theorem is to develop the necessary technics to compute certain pushouts in the category $\mathbf{Cat_{Top}}$ based on the work of Dwyer and Kan [4] and Lurie [9]. Roughly speaking, sections 2, 3 and 4 are the technical heart of the complete proof of the main theorem. Section 2 is devoted to an explicit construction of free topological categories.
In section 3, we introduce a notion of good realization of simplicial object in the category of topological categories with a fixed set $\mathcal{O}$ of objects. The main problem is due to the fact that the objects in **Top** are not cofibrant in general.

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Section 4 is the technical part of the article. We introduce the notion of free resolution of topological categories defined initially in [4] and we compute explicitly some push-outs in $\mathbf{Cat_{Top}}$.
Section 5 and 6 is a relation between the model categories of the Joyal model structure, Bergner model structure, the new model structure on $\mathbf{Cat_{Top}}$ and the standard model structure on simplicial sets. We study precisely the relation of different notions of $\infty$-groupoids in each model and establish relation between them in order to prepare for the description of the mapping space of the model categories $\mathbf{Cat_{Top}}$ and $\mathbf{Cat_{sSet}}$.
In section 7, we give a model for the mapping spaces of the category $\mathbf{Cat_{Top}}$ equivalent to those described by Toën in [11] but equivalent. As consequence we obtain very interesting and nontrivial equivalences between mapping spaces. Finally, section 8 is about a functorial construction of Dwyer-Kan localization with nice properties. The article is self contained.

**Theorem 1.3** *The category of small topological categories is a cofibratly generated model category where the weak equivalences are the Dwyer-Kan equivalences. It is Quillen equivalent to the Bergner model structure on the category of small simplicial categories.*

The categories $\mathbf{Cat_{Top}}$ and $\mathbf{Cat_{sSet}}$ are **not** symmetric monoidal model categories but following the ideas of the article [11] we have as a direct consequence that the homotopy categories $\mathrm{Ho}\mathbf{Cat_{Top}}$ and $\mathrm{Ho}\mathbf{Cat_{sSet}}$ are symmetric monoidal **closed** categories. The tensor product is denoted by $\otimes^{\mathbb{L}}$ and the internal hom by $\mathbb{R}\mathbf{HOM}$. The enriched category $\mathbb{R}\mathbf{HOM}(\mathbf{C},\mathbf{D})$ is the full sub-category of right quasi representable $\mathbf{C}^{op}\otimes^{\mathbb{L}}\mathbf{D}$-Modules.
**Theorem 7.3** *Let* $\mathbf{D}$ *be a topological category or fibrant simplicial category and* $\mathbf{C}$ *a cofibrant object in* $\mathbf{Cat_V}$ *then:*

$$\mathbf{map}_{\mathbf{Cat_V}}(\mathbf{C},\mathbf{D}) \sim \mathrm{N}_\bullet\ w\mathbb{R}\mathbf{HOM}(\mathbf{C},\mathbf{D}) \sim \widetilde{\mathrm{N}_\bullet} G\mathbb{R}\mathbf{HOM}(\mathbf{C},\mathbf{D})$$

*where* $G\mathbb{R}\mathbf{HOM}(\mathbf{C},\mathbf{D})$ *the* $\infty$*-groupoid associated to* $\mathbb{R}\mathbf{HOM}(\mathbf{C},\mathbf{D})$ *,* $w\mathbb{R}\mathbf{HOM}(\mathbf{C},\mathbf{D})$ *is the* ***discrete*** *subcategory of weak equivalences,* $\mathrm{N}_\bullet$ *the standard nerve of discrete categories,* $\widetilde{\mathrm{N}_\bullet}$ *is the coherent nerve and* $\mathbf{V}=\mathbf{sSet}$ *or* $\mathbf{Top}$*. More specially*

$$\widetilde{\mathrm{N}_\bullet} G\mathbf{D} \sim \mathrm{N}_\bullet w\mathbb{R}\mathbf{HOM}(*,\mathbf{D}) \sim \widetilde{\mathrm{N}_\bullet} G\mathbb{R}\mathbf{HOM}(*,\mathbf{D}).$$

We also prove the following proposition which is, in some sense, a justification for the homotopy invariance of algebraic $\mathcal{K}$-theory of a suitable model categories of modules. This theorem which also connects the discrete case to the enriched one.
**Corollary 7.4** *Suppose that* $\mathbf{D}=\mathbb{R}\mathbf{HOM}(*,\mathbf{C})$ *then*

$$\widetilde{\mathrm{N}_\bullet} G\mathbf{D} \sim \mathrm{N}_\bullet\ w\mathbf{D}.$$

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## 1. Category of small topological categories.

In this article, the category of Hausdorf compactly generated topological spaces will be denoted by **Top**. it is a simplicial symmetric monoidal closed model category.
Before stating the main theorem of this section we will introduce some notations and definitions.

**Remark 1.1.** A topological category is a category enriched over **Top**. The category of all (small) topological categories is denoted by $\mathbf{Cat}_{\mathbf{Top}}$. The morphisms in $\mathbf{Cat}_{\mathbf{Top}}$ are the enriched functors. It is a complete and cocomplete category.
The category **V** will be either the category **sSet** or **Top**.

**Definition 1.2.** Let $\mathcal{U} : \mathbf{V} \to \mathbf{Cat}_{\mathbf{V}}$ be the functor defined as follows:
For each object $S \in \mathbf{V}$, $\mathcal{U}(S)$ is the enriched category with two objects $x$ and $y$ such that $\mathbf{Map}_{\mathcal{U}(S)}(x, y) = S$.

**Theorem 1.3.** [1] *The category* $\mathbf{Cat}_{\mathbf{Top}}$ *admits a cofibranty generated model structure defined as follows.*
*The weak equivalences* $\mathrm{F} : \mathbf{C} \to \mathbf{D}$ *satisfy the following conditions.*

WT1 : *The morphism* $\mathbf{Map}_{\mathbf{C}}(a, b) \to \mathbf{Map}_{\mathbf{D}}(\mathrm{F}a, \mathrm{F}b)$ *is a weak equivalence in the category* **Top**.
WT2 : *The induced morphism* $\pi_0\mathrm{F} : \pi_0\mathbf{C} \to \pi_0\mathbf{D}$ *is a categorical equivalence in* **Cat**.

*The fibrations are the morphisms* $\mathrm{F} : \mathbf{C} \to \mathbf{D}$ *which satisfy :*

FT1 : *The morphism* $\mathbf{Map}_{\mathbf{C}}(a, b) \to \mathbf{Map}_{\mathbf{D}}(\mathrm{F}a, \mathrm{F}b)$ *is a fibration in* **Top**.
FT2 : *For each objects $a$ and $b$ in* $\mathbf{C}$, *and a weak equivalence* $e : \mathrm{F}(a) \to b$ *in* $\mathbf{D}$ *(i.e., an isomorphism in* $\pi_0\mathbf{D}$*), there exists an object* $a_1$ *in* $\mathbf{C}$ *and a weak equivalence* $d : a \to a_1$ *in* $\mathbf{C}$ *such that* $\mathrm{F}d = e$.

*Moreover, the set $I$ of generating cofibrations is given by :*

CT1 : $|\mathcal{U}\partial\Delta^n| \to |\mathcal{U}\Delta^n|$, *for* $n \geq 0$.
CT1 : $\emptyset \to \{x\}$, *where* $\emptyset$ *is the empty topological category and* $\{x\}$ *is the category with one object and one morphism.*

*The set $J$ of generating acyclic cofibrations is given by:*

ACT1 : $|\mathcal{U}\Lambda^n_i| \to |\mathcal{U}\Delta^n|$, *for* $0 \leq n$ *and* $0 \leq i \leq n$.
ACT2 : $\{x\} \to |\mathcal{H}|$ *where* $\{\mathcal{H}\}$ *is defined in* [2].

Moreover, the adjunction:

$$\mathbf{Cat}_{\mathbf{sSet}} \underset{\text{sing}}{\overset{|-|}{\rightleftarrows}} \mathbf{Cat}_{\mathbf{Top}}$$

is a Quillen equivalence.

**Remark 1.4.** Obviously, all objects in $\mathbf{Cat}_{\mathbf{Top}}$ are *fibrant.*

We start with a useful lemma which gives us conditions to transfer a model structure by adjunction.

**Lemma 1.5.** *[[12], proposition 3.4.1] Consider an adjunction*

$$\mathbf{M} \underset{F}{\overset{G}{\rightleftarrows}} \mathbf{C}$$

*where $\mathbf{M}$ is a cofibrantly generated model category, with generating cofibrations $\mathrm{I}$ and generating trivial cofibrations $\mathrm{J}$. We pose*

- $\mathrm{W}$ *the class of morphisms in $\mathbf{C}$ such the image by $F$ is a weak equivalence in $\mathbf{M}$.*
- $\mathrm{F}$ *the class of morphisms in $\mathbf{C}$ such the image by $F$ is a fibration in $\mathbf{M}$.*

*We suppose that the following conditions are verified:*

(1) *The domains of $G(i)$ are small with respect to $G(\mathrm{I})$ for all $i \in \mathrm{I}$ and the domains of $G(j)$ are small with respect to $G(\mathrm{J})$ for all $j \in \mathrm{J}$.*
(2) *The functor $F$ commutes with directed colimits i.e.,*

$$F\mathrm{colim}(\lambda \to \mathbf{C}) = \mathrm{colim}F(\lambda \to \mathbf{C}).$$

(3) *Every transfinite composition of weak equivalences in $\mathbf{M}$ is a weak equivalence.*
(4) *The pushout of $G(j)$ by any morphism $f$ in $\mathbf{C}$ is in $\mathrm{W}$.*

*Then $\mathbf{C}$ forms a model category with weak equivalences (resp. fibrations) $\mathrm{W}$ (resp. $\mathrm{F}$). Moreover, it is cofibrantly generated with generating cofibrations $G(\mathrm{I})$ and generating trivial cofibrations $G(\mathrm{J})$.*

We prove the main theorem using 1.5.

**Lemma 1.6.** *The pushout of $|U\Lambda^n_i| \to |U\Delta^n|$ along any morphism $\mathrm{F} : |U\Lambda^n_i| \to \mathbf{D}$ is a weak equivalence.*

*Proof.* See 4.4 □

**Lemma 1.7.** *The pushout of $\{x\} \to |\mathcal{H}|$ along $\{x\} \to \mathbf{C}$ is a weak equivalence for all $\mathbf{C} \in \mathbf{Cat_{Top}}$.*

*Proof.* Let $\mathcal{O}$ be the set of objects of $\mathbf{C}$ without the object $\{x\}$ touched by the morphism $\{x\} \to \mathbf{C}$. We note by $x,\ y$ objects of $|\mathcal{H}|$. The goal is to prove that $h$ defined in the following pushout is a weak equivalence

$$\begin{array}{ccc} \{x\} & \longrightarrow & \mathbf{C} \\ \big\downarrow & & \big\downarrow{\scriptstyle h} \\ |\mathcal{H}| & \longrightarrow & \mathbf{D} \end{array}$$

Observe that there is an other double pushout

$$\begin{array}{ccc} \{x\} \sqcup \mathcal{O} & \longrightarrow & \mathbf{C} \\ \big\downarrow & & \big\downarrow{\scriptstyle i} \\ \{x,y\} \sqcup \mathcal{O} & \longrightarrow & \mathbf{C} \sqcup \{y\} \\ \big\downarrow & & \big\downarrow{\scriptstyle h'} \\ |\mathcal{H}| \sqcup \mathcal{O} & \longrightarrow & \mathbf{D}. \end{array}$$

Which is a consequence of:

$$|\mathcal{H}| \sqcup \mathcal{O} \bigsqcup_{\mathcal{O} \sqcup \{x,y\}} \mathbf{C} \sqcup \{y\} = |\mathcal{H}| \bigsqcup_{\{x,y\}} \mathbf{C} \sqcup \{y\} = |\mathcal{H}| \bigsqcup_{\{x\}} \mathbf{C} = \mathbf{D}.$$

The morphism $h^{'}$ is a natural extension of $h$, i.e., $h^{'} \circ i = h$.

On the other hand, the counity $c : |\text{sing}\mathbf{C}| \to \mathbf{C}$ is a weak equivalence. Consider the following pushout in $\mathbf{Cat_{sSet}}$:

$$\begin{array}{ccc}
\{x\} \sqcup \mathcal{O} & \longrightarrow & \text{sing}\mathbf{C} \\
\downarrow & & \downarrow i \\
\{x,y\} \sqcup \mathcal{O} & \longrightarrow & \text{sing}(\mathbf{C}) \sqcup \{y\} \\
\downarrow & & f^{'} \downarrow \\
\mathcal{H} \sqcup \mathcal{O} & \longrightarrow & \mathbf{D}^{'}.
\end{array}$$

Since $\mathbf{Cat_{sSet}}$ is a model category, we have that $f = f^{'} \circ i$ is a weak equivalence. Consequently $|f|$ is a weak equivalence in $\mathbf{Cat_{Top}}$ .
As before $f^{'}$ is an extension of $f$.

Using the fact that the functor $| - |$ commutes with colimits, the diagram of the following double pushout permit to conclude:

$$\begin{array}{ccccc}
 & & |\text{sing}\mathbf{C}| & \xrightarrow{\sim} & \mathbf{C} \\
 & & \downarrow i & & \downarrow i \\
\{x,y\} \sqcup \mathcal{O} & \longrightarrow & |\text{sing}(\mathbf{C} \sqcup \{y\})| & \xrightarrow{c} & \mathbf{C} \sqcup \{y\} \\
\downarrow & & |f^{'}| \downarrow & & h^{'} \downarrow \\
|\mathcal{H}| \sqcup \mathcal{O} & \longrightarrow & |\mathbf{D}^{'}| & \xrightarrow[m]{} & \mathbf{D}.
\end{array}$$

In Fact,

$$m : \mathbf{D} = (|\mathcal{H}| \sqcup \mathcal{O}) \star |\text{sing}(\mathbf{C} \sqcup \{y\})| \to (|\mathcal{H}| \sqcup \mathcal{O}) \star (\mathbf{C} \sqcup \{y\}) = \mathbf{D}^{'}$$

is a weak equivalence by 4.7. We have seen that $|f|$ is a weak equivalence, so by the property *2 out of 3* we conclude that $h$ is a weak equivalence. □

**Lemma 1.8.** *The functor* sing *commutes with directed colimits.*

*Proof.* Let $\lambda$ be an ordinal and let

$$\mathbf{C} = \text{colim}_\lambda \mathbf{C}_\lambda,$$

a directed colimit in $\mathbf{Cat_{Top}}$. If $a^{'}$ and $b^{'}$ are two objects in $\mathbf{C}$, then by definition, there exists an index $t$ such that they are represented by $a,\ b \in \mathbf{C}_t$, and $\mathbf{Map_C}(a^{'}, b^{'})$ is a colimit of the following diagram:

$$\mathbf{Map}_{\mathbf{C}_t^{a,b}}(a, b) \to \ldots \mathbf{Map}_{\mathbf{C}_s}(a_s, b_s) \to \mathbf{Map}_{\mathbf{C}_{s+1}}(a_{s+1}, b_{s+1}) \to \ldots$$

where $\mathbf{C}_t^{a,b}$ is a full subcategory of $\mathbf{C}_t$ with only two objects $a,\ b$. Since the functor Ob : $\mathbf{Cat} \to \mathbf{Set}$ and the functor sing : $\mathbf{Top} \to \mathbf{sSet}$ commute with directed colimits, we have that sing : $\mathbf{Cat_{Top}} \to \mathbf{Cat_{sSet}}$ commutes with directed colimits. □

**Lemma 1.9.** *The objects* $|\mathcal{U}\Lambda_i^n|$, $|\mathcal{U}\Delta^n|$ *and* $|\mathcal{H}|$ *are small in* $\mathbf{Cat_{Top}}$

*Proof.* It is a consequence of the fact that $\mathcal{U}\Lambda_i^n$, $\mathcal{U}\Delta^n$ , $\mathcal{H}$ are small in $\mathbf{Cat_{sSet}}$ and sing : $\mathbf{Cat_{Top}} \to \mathbf{Cat_{sSet}}$ commutes with directed colimits. □

**Lemma 1.10.** *The transfinite composition of weak equivalences in* $\mathbf{Cat_{sSet}}$ *is a weak equivalences.*

*Proof.* It is a consequence that the transfinite composition of weak equivalences in **sSet** and **Cat** is a weak equivalence. Note that $\pi_0 : \mathbf{Cat_{sSet}} \to \mathbf{Cat}$ commutes with colimits because it admits a right adjoint: the functor which correspond to each topological enriched category **C** an trivially enriched category i.e., we forget the topology of **C**. □

**Corollary 1.11.** *The category* $\mathbf{Cat_{Top}}$ *is a cofibrantly generated model category Quillen equivalent to* $\mathbf{Cat_{sSet}}$.

***Proof of the main theorem****1.3.* It follows from lemma 1.5. The point (1) is proven by 1.9. The point (2) is proven by 1.8. The point (3) is proven by 1.10 and finally the point (4) is given by 1.6, 1.7. The Quillen equivalence between $\mathbf{Cat_{Top}}$ and $\mathbf{Cat_{sSet}}$ is a direct consequence of the Quillen equivalence between **Top** and **sSet**.

□

## 2. Free Enriched Categories

In this section, we define an adjunction between $\mathbf{Cat_{Top}}$ and the categories of enriched graphs on **Top**. This adjunction is constructed in the particular case where the set of objects is fixed. We will denote $\mathcal{O} - \mathbf{Cat_{Top}}$ the category of small enriched categories over **Top** with fixed set of objects $\mathcal{O}$, the morphisms are those functors which are identities on objects. By the same way, we define the category $\mathcal{O} - \mathbf{Graph_{Top}}$ of small graphs enriched over **Top** with a fixed set of vertices $\mathcal{O}$. There exists an adjunction between $\mathcal{O} - \mathbf{Cat_{Top}}$ and $\mathcal{O} - \mathbf{Graph_{Top}}$ given by the forgetful functor and the free functor. First of all, we define the free functor between graphs and categories. First we study the case where $\mathcal{O}$ is a set with one element.

**Lemma 2.1.** *There exists a right adjoint to the forgetful functor* $U : \mathbf{Mon} \to \mathbf{Top}$ *where* **Mon** *is the category of topological monoids.*

*Proof.* Let $X$ be a space. We define

$$L(X) = * \sqcup X \sqcup (X \times X) \sqcup (X \times X \times X) \sqcup \dots;$$

it is a a well defined functor from **Top** to topological monoids.

In fact, it is the desired functor. Let $M$ be a topological monoid, a morphism of monoids $L(X) \to M$ is given by a morphism of non pointed topological spaces $X \to U(M)$. This morphism extends in a unique way to a morphism of monoids if we consider the following morphisms in **Top**:

$$X \times X \cdots \times X \to M \times M \cdots \times M \to M.$$

We conclude that: $\mathbf{hom_{Top}}(X, U(M)) = \mathbf{hom_{Mon}}(L(X), M)$. □

For a generalization to an adjunction between $\mathcal{O} - \mathbf{Cat_{Top}}$ and $\mathcal{O} - \mathbf{Graph_{Top}}$, we do as follow: we pose **O** the trivial category with set of object $\mathcal{O}$. For each graph $\mathbf{\Gamma}$ in $\mathcal{O}-\mathbf{Graph_{Top}}$ we define the set of the following categories indexed by a pair of element $a,\ b \in \mathcal{O}$

$$\mathbf{\Gamma}_{a,b}(c,d) = \begin{cases} \mathbf{\Gamma}(c,d) & \text{if } c = a \neq b = d \\ L(\mathbf{\Gamma}(c,d)) & \text{if } a = c = b = d \\ \emptyset & \text{if } c \neq d \text{ and } a \neq c \wedge b \neq d \\ * = id & \text{else} \end{cases}$$

Let $\mathbf{\Gamma}$ be a graph in $\mathcal{O}-\mathbf{Graph_{Top}}$. we define the free category induced by the graph as a free product in the category $\mathcal{O}-\mathbf{Cat_{Top}}$ of all categories of the form $\mathbf{\Gamma}_{a,b}$, more precisely

$$L(\mathbf{\Gamma}) = \star_{(a,b)\in\mathcal{O}\times\mathcal{O}}\mathbf{\Gamma}_{a,b}.$$

By the free product, we mean the following colimit in $\mathbf{Cat_{Top}}$:

$$\mathrm{colim}_{(a,b)\in\mathcal{O}\times\mathcal{O}}\mathbf{\Gamma}_{a,b}.$$

**Lemma 2.2.** *We have a generalized adjunction:*

$$\mathcal{O}-\mathbf{Graph_{Top}} \underset{U}{\overset{L}{\rightleftarrows}} \mathcal{O}-\mathbf{Cat_{Top}}$$

*Proof.* It follows directly from the construction of the extended (free) functor $L$. $\square$

## 3. Realization

Let $\mathbf{M}$ be a simplicial model category (i.e., tensored and cotensored in a suitable way [6]). The category $[\Delta^{op}, \mathbf{M}]$ is a model category with Reedy model structure (cf [6]) where the weak equivalences are defined degree wise.

**Definition 3.1.** The realization functor

$$|-| : [\Delta^{op}, \mathbf{M}] \to \mathbf{M}$$

is defined as follows:

$$\bigsqcup_{\phi:[n]\to[m]} M_m \otimes \Delta^n \underset{d_1}{\overset{d_0}{\rightrightarrows}} \bigsqcup_{[n]} M_n \otimes \Delta^n \longrightarrow |M_\bullet|$$

where $d_0 = \phi^* \otimes id$ and $d_1 = id \otimes \phi$. It is a coequalizer.

**Remark 3.2.** Since $\mathbf{M}$ is a simplicial category, the functor $|-|$ admits a right adjoint:

$$(-)^\Delta : \mathbf{M} \to [\Delta^{op}, \mathbf{M}] : M \mapsto M^{\Delta^n}.$$

**Lemma 3.3.** *[*[6]*, VII, proposition 3.6] Let* $\mathbf{M}$ *be a simplicial model category and* $[\Delta^{op}, \mathbf{M}]$ *equipped with the Reedy model structure. Then the realization functor*

$$|-| : [\Delta^{op}, \mathbf{M}] \to \mathbf{M}$$

*is a left Quillen functor.*

Now, we specialize to $\mathbf{M} = \mathbf{Top}$. In this particular case, $[\Delta^{op}, \mathbf{Top}]$ is a monoidal category (the monoidal structure is defined degree wise form the monoidal structure of $\mathbf{Top}$). So, the realization functor $|-| : [\Delta^{op}, \mathbf{Top}] \to \mathbf{Top}$ commutes with the monoidal product (cf [5], chapter X, proposition 1.3).

**Corollary 3.4.** *The realization functor* $|-| : [\Delta^{op}, \mathbf{Top}] \to \mathbf{Top}$ *preserves homotopy equivalences.*

In practice, lemma 3.3 is difficult to use. It is quite difficult to show that an object in $[\Delta^{op}, \mathbf{M}]$ is Reedy cofibrant. In appendix A of [10], Segal gives us an alternative solution in the particular case of $[\Delta^{op}, \mathbf{Top}]$.

**Lemma 3.5.** *There exists a functor* $||-|| : [\Delta^{op}, \mathbf{Top}] \to \mathbf{Top}$*, called good realization with the following properties:*

(1) *Let* $f_\bullet : X_\bullet \to Y_\bullet$ *be a morphism in* $[\Delta^{op}, \mathbf{Top}]$ *such that if* $f_n : X_n \to Y_n$ *is a weak equivalence for all* $n \in \mathbb{N}$*, then* $||f_\bullet|| : ||X_\bullet|| \to ||Y_\bullet||$ *is a weak equivalence in* **Top***;*
(2) *There exists a natural transformation* $\mathcal{N} : ||-|| \to |-|$*, with the property that for all* ***good simplicial topological space*** $X_\bullet$*, the natural morphism:*

$$\mathcal{N}_{X_\bullet} : ||X_\bullet|| \to |X_\bullet|$$

*is a weak equivalence in* **Top***;*
(3) *The natural morphism* $||X_\bullet \times Y_\bullet|| \to ||X_\bullet|| \times ||Y_\bullet||$ *is a weak equivalence in* **Top**.

For the details we refer to [10].

**Lemma 3.6.** [10] *There exists an endofunctor* $\tau : [\Delta^{op}, \mathbf{Top}] \to [\Delta^{op}, \mathbf{Top}]$ *and a natural transformation* $\mathcal{Q} : \tau \to id$ *with the following properties:*

(1) $\tau X_\bullet$ *is a good simplicial topological space for all* $X_\bullet \in [\Delta^{op}, \mathbf{Top}]$*;*
(2) *The natural morphism* $\mathcal{Q}_n : \tau_n(X_\bullet) \to X_n$ *is a weak equivalence for all* $n \in \mathbb{N}$*;*
(3) *The natural morphism* $||X_\bullet|| \to |\tau(X_\bullet)|$ *is a weak equivalence;*
(4) *Finally , we have* $\tau_0(X_\bullet) = X_0$.

**Corollary 3.7.** *Let* $f_\bullet : X_\bullet \to Y_\bullet$ *be a morphism in* $[\Delta^{op}, \mathbf{Top}]$*, such that* $f_n$ *is a weak equivalence for all* $n$*, then*

$$|\tau(f_\bullet)| : |\tau(X_\bullet)| \to |\tau(Y_\bullet)|$$

*is a weak equivalence of topological spaces.*

*Proof.* It is a direct consequence from 3.5 and 3.6. □

We can see the functor $\tau$ as kind of cofibrant replacement. It is useful to know how to describe the functor $\tau$.

**Definition 3.8.** [[10], appendix A] Let $A_\bullet$ be a simplicial topological space and $\sigma$ a subset of $\{1, \dots, n\}$. We pose:

(1) $A_{n,i} = s_i A_n$.
(2) $A_{n,\sigma} = \cap_{i \in \sigma} A_{n,i}$.
(3) $\tau_n(A_\bullet)$ is a union of all subsets $[0,1]^\sigma \times A_{n,\sigma}$ of $[0,1]^n \times A_n$.

The morphism $\tau(A_\bullet) \to A_\bullet$ collapses $[0,1]^\sigma$ and injects $A_{n,\sigma}$ in $A_n$.

**Lemma 3.9.** *The functor* $\tau$ *sends homotopy equivalences to homotopy equivalences.*

*Proof.* Let $h : X_\bullet \times [0,1] \to Y_\bullet$ be a homotopy between $t$ and $s$. By definition of $\tau$, we have

$$\begin{array}{rcl} \tau_n(X_\bullet \times [0,1]) & = & \bigcup_{\sigma \in \{1,\dots n\}} [0,1]^\sigma \times (X_\bullet \times [0,1])_{n,\sigma} \\ & = & \bigcup_{\sigma \in \{1,\dots n\}} ([0,1]^\sigma \times X_{n,\sigma} \times [0,1]) \\ & = & (\bigcup_{\sigma \in \{1,\dots n\}} [0,1]^\sigma \times X_{n,\sigma}) \times [0,1] \\ & = & \tau_n(X_\bullet) \times [0,1]. \end{array}$$

Consequently $\tau(h) : \tau(X_\bullet) \times [0,1] \to \tau(Y_\bullet)$ is a homotopy between $\tau(t)$ and $\tau(s)$. □

**Definition 3.10.** A strong section $f : X \to Y$ is a continuous map $i : Y \to X$ such that $f \circ i = id_Y$ and such that there exists a homotopy between $i \circ f$ and $id_X$ which fixe $Y$.

**Corollary 3.11.** *The functor $\tau$ preserve strong sections.*

*Proof.* It is a consequence of lemma 3.9 and the fact that $\tau$ is a functor so it preserves the identities. □

**Corollary 3.12.** *If $X$ is a constant simplicial topological space, then $\mathcal{Q}_X : \tau(X) \to X$ admits a strong section.*

*Proof.* The section $i : X \to \tau(X)$ is induced by the identity on $X$. To show that it is a strong section, it is sufficient to see that $\tau_n(X) = [0,1]^n \times X$ by definition. □

## 4. Push-outs in $\mathbf{Cat_V}$

We define and compute some (simple) pushouts in the category of small enriched categories $\mathbf{Cat_V}$. In our example $\mathbf{V}$ is the category **sSet** or **Top**. For more details see ([9], A.3.2).

Let $f : S \to T$ be a morphism in $\mathbf{V}$ and $\mathbf{C}$ an enriched category on $\mathbf{V}$. We want to describe explicitly the following pushout diagram:

$$\begin{array}{ccc} \mathcal{U}S & \xrightarrow{h} & \mathbf{C} \\ \downarrow{\scriptstyle \mathcal{U}f} & & \downarrow \\ \mathcal{U}T & \longrightarrow & \mathbf{D} \end{array}$$

It is enough clear that the objects of $\mathbf{C}$ and $\mathbf{D}$ are the same. The difficult par is to define $\mathbf{Map_D}$.

Let $w,\ z \in \mathbf{C}$ and define the following sequence of objets in $\mathbf{V}$:

$$\begin{array}{rcl} M^0_{\mathbf{C}} & = & \mathbf{Map_C}(w,z). \\ M^1_{\mathbf{C}} & = & \mathbf{Map_C}(y,z) \times T \times \mathbf{Map_C}(w,x). \\ M^2_{\mathbf{C}} & = & \mathbf{Map_C}(y,z) \times T \times \mathbf{Map_C}(y,x) \times T \times \mathbf{Map_C}(w,x). \\ \dots & & \end{array}$$

More generally, an object of $M^k_{\mathbf{C}}$ is given by a finite sequence of the form

$$(\sigma_0, \tau_1, \sigma_1, \tau_2, \dots, \tau_k, \sigma_k)$$

where

$$\sigma_0 \in \mathbf{Map_C}(y,z),\ \sigma_k \in \mathbf{Map_C}(w,x),\ \sigma_i \in \mathbf{Map}(y,x)$$

for $0 < i < k$, and $\tau_i \in T$ for $0 < i \leq k$.

We define $\mathbf{Map_D}(w,z)$ as a quotient of $\coprod_k M^k_{\mathbf{C}}$ relative to the following relations:

$$(\sigma_0, \tau_1, \dots, \sigma_k) \sim (\sigma_0, \tau_1, \dots, \tau_{j-1}, \sigma_{j-1} \circ h(\tau_j) \circ \sigma_j, \tau_{j+1}, \dots, \sigma_k),$$

when $\tau_j$ is an element of $S \subset T$.

The category $\mathbf{D}$ is equipped with the following associative composition:

$$(\sigma_0, \tau_1, \dots, \sigma_k) \circ (\sigma_0^{'}, \tau_1^{'}, \dots, , \sigma_l^{'}) = (\sigma_0, \tau_1, \dots, , \tau_k, \sigma_k \circ \sigma_0^{'}, \tau_1^{1}, \dots, \sigma_l^{'}).$$

Observe that there is a natural filtration on $\mathbf{Map_D}(w,z)$:

$$\mathbf{Map_C}(w,z) = \mathbf{Map}_D(w,z)^0 \subset \mathbf{Map_D}(w,z)^1 \subset \dots$$

where $\mathbf{Map}_D(w,z)^k$ is defined as image of $\coprod_{0 \leq i \leq k} M^i_{\mathbf{C}}$ in $\mathbf{Map_D}(w,z)$ and

$$\bigcup_k \mathbf{Map_D}(w,z)^k = \mathbf{Map}_D(w,z).$$

The most important fact is that $\mathbf{Map_D}(w,z)^k \subset \mathbf{Map_D}(w,z)^{k+1}$ is constructed as pushout of the inclusion: $N^{k+1}_{\mathbf{C}} \subset M^{k+1}_{\mathbf{C}}$, where $N^{k+1}_{\mathbf{C}}$ is a sub-object of $M^{k+1}_{\mathbf{C}}$ of $(2m+1)$-tuples $(\sigma_0, \tau_1, ..., \sigma_m)$ such that $\tau_i \in S$ for at less one $i$.

4.1. **Monads.** The main goal of this section is to generalize the section 2 of the article [4] to the categories enriched over **Top**.
Every adjunction define a monad and a comonad. We are interested on the particular adjunction 2.2

$$\mathcal{O} - \mathbf{Graph_{Top}} \underset{U}{\overset{L}{\rightleftarrows}} \mathcal{O} - \mathbf{Cat_{Top}}$$

We have a monad $T = UL$ and a comonad $F = LU$. The multiplication on $T$ is denoted by $\mu : TT \to T$ and the unity $\eta : id \to T$, the comultiplication by $\psi : F \to FF$ and finally the counity by $\phi : F \to id$. The $T-$algebras are exactly those graphs which have a structure of a category (composition).

**Notation 4.1.** We denote by $\mathcal{O} - \mathbf{sCat_{Top}}$ the category of presheaves $[\Delta^{op}, \mathcal{O} - \mathbf{Cat_{Top}}]$ and
$\mathcal{O} - \mathbf{sGraph_{Top}}$ the category of prescheaves $[\Delta^{op}, \mathcal{O} - \mathbf{Graph_{Top}}]$.
If we note $[\Delta^{op}, \mathbf{Top}]$ by $\mathbf{sTop}$ then we have

$$\mathcal{O} - \mathbf{sCat_{Top}} = \mathcal{O} - \mathbf{Cat_{sTop}},$$

and

$$\mathcal{O} - \mathbf{sGraph_{Top}} = \mathcal{O} - \mathbf{Graph_{sTop}}.$$

4.1.1. *Simplicial resolution.* Let $\mathbf{C}$ be an object of $\mathcal{O}-\mathbf{Cat_{Top}}$,

We define the iterated composition of $F$ by :

$$F^k = \underbrace{F \circ F \cdots \circ F}_{k}.$$

The comonad $F$ gives us a simplicial resolution $\mathbf{C}$ (cf [4]) defined as follow:

$$F_k\mathbf{C} = F^{k+1}\mathbf{C},$$

With faces and degeneracies:

$$F_k\mathbf{C} \xrightarrow{d_i = F^i \phi F^{k-i}} F_{k-1}\mathbf{C}$$

$$F_k\mathbf{C} \xrightarrow{s_i = F^i \psi F^{k-i}} F_{k+1}\mathbf{C}$$

The category of compactly generated spaces $\mathbf{Top}$ is a simplicial model category (tensored and cotensored over $\mathbf{sSet}$):

(1) In $\mathcal{O}-\mathbf{sCat_{Top}}$ we have the morphism $f : F_\bullet\mathbf{C} \to \mathbf{C}$, where $\mathbf{C}$ is sow as a constant object in $\mathcal{O}-\mathbf{sCat_{Top}}$ and t $f_k = \phi^{k+1}$.
(2) The morphism $f$ admits a section $i : \mathbf{C} \to F_\bullet\mathbf{C}$ in the category $\mathbf{Graph_{sTop}}$. The section $i$ is induced by the unit of the monad $T$ i.e., $\eta_{U\mathbf{C}} : U\mathbf{C} \to ULU\mathbf{C}$;
(3) The adjunction

$$[\Delta^{op}, \mathbf{Top}] \underset{(-)^{\Delta}}{\overset{|-|}{\rightleftarrows}} \mathbf{Top},$$

induces the following adjunction

$$\mathcal{O}-\mathbf{Cat_{sTop}} \underset{(-)^{\Delta}}{\overset{|-|}{\rightleftarrows}} \mathcal{O}-\mathbf{Cat_{Top}},$$

since the realization functor is monoidal.
(4) The realization of the morphism $f$ in $\mathcal{O}-\mathbf{sCat_{Top}}$ induces a weak equivalence i.e., $|f| : \mathbf{Map}_{|F_\bullet\mathbf{C}|}(a,b) \to \mathbf{Map_C}(a,b)$ is a weak equivalence in $\mathbf{Top}$ for all $a, b \in \mathcal{O}$.

**Remark 4.2.** The realization functor $|-|$ does not "see" the category structure, but only the graph structure.

More generally, for all $\mathbf{C},\ \mathbf{D}$ in $\mathcal{O}-\mathbf{Cat_{Top}}$ the following morphism:

$$F_\bullet(\mathbf{C}) \star \mathbf{D} \longrightarrow \mathbf{C} \star \mathbf{D}$$

admits a strong section $\mathbf{C}\star\mathbf{D} \to F_\bullet\mathbf{C}\star\mathbf{D}$ in the category $\mathcal{O}-\mathbf{sGraph_{Top}}$. In fact, the category $\mathcal{O}-\mathbf{Graph_{Top}}$ is monoidal (nonsymmetric) with monoidal product $\times_{\mathcal{O}}$ which is a generalization of ([8],II, 7). A topologically enriched category is a monoid with respect to this monoidal product. The free product $\mathbf{C} \star \mathbf{D}$ is constructed in $\mathcal{O}-\mathbf{Graph_{Top}}$ as

$$\mathcal{O}^c \sqcup_{\mathcal{O}} \mathbf{C}^{'} \sqcup_{\mathcal{O}} \mathbf{D}^{'} \sqcup_{\mathcal{O}} (\mathbf{C}^{'} \times_{\mathcal{O}} \mathbf{C}^{'}) \sqcup_{\mathcal{O}} (\mathbf{D}^{'} \times_{\mathcal{O}} \mathbf{D}^{'}) \sqcup_{\mathcal{O}} (\mathbf{C}^{'} \times_{\mathcal{O}} \mathbf{D}^{'}) \sqcup_{\mathcal{O}} (\mathbf{D}^{'} \times_{\mathcal{O}} \mathbf{C}^{'}) \ldots$$

where $\mathbf{C}^{'}$ (resp. $\mathbf{D}^{'}$) is a correspondent graph of $\mathbf{C}$ (resp. $\mathbf{D}$) without identities and $\mathcal{O}^c$ is the trivial category obtained from the set $\mathcal{O}$. So $\mathbf{C}\star\mathbf{D}\to F_{\bullet}(\mathbf{C})\star\mathbf{D}$ is induced by the section $i:\mathbf{C}\to F_{\bullet}\mathbf{C}$ and $id:\mathbf{D}\to\mathbf{D}$. consequently the morphism

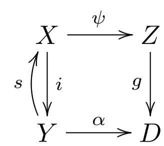
$$\mathbf{Map}_{\mathbf{C}\star\mathbf{D}}(a,b)\to\mathbf{Map}_{|F_i(\mathbf{C})\star\mathbf{D}|}(a,b)=\mathbf{Map}_{|F_i(\mathbf{C})|\star\mathbf{D}}(a,b)$$

is a weak equivalence in **Top** for all objects $a,\ b\in\mathcal{O}$.

**Lemma 4.3.** *Let $i:\ X\to Y$ an inclusion and a weak equivalence of topological spaces and $i(X)$ closed in $Y$ such that there exists a homotopy $H:Y\times[0,1]\to Y$ which verify the following conditions:*

1. $H(-,0)=id_Y$
2. $H(i(x),t)=i(x)$ *for all* $x\in X$.
3. $H(-,1)=s$ *with* $s\circ i=id_X$.

*then the morphism $g$ of the pushout :*



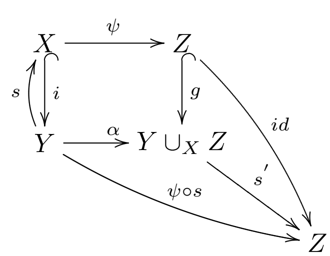

*is a weak equivalence.*

*Proof.* We remind that $D=Y\cup_X Z$. To simplify notation we will denote the image of $y\in Y$ in $D$ by $y$, respectively $z$ for the image of $z\in Z$ in $D$ .

Since $i$ admits a retraction, $g$ admits also a retraction noted by $s^{'}$ and induced by $s$. It means that we have an inclusion of $Z$ in $D$ via $g$ because of $s^{'}\circ g=id_Z$. In fact, $s^{'}:D\to Z$ is defined as follow:

1. $s^{'}(z)=z$ for $z\in Z$.
2. $s^{'}(y)=s(y)$ for $y\in Y$.

This new section $s^{'}$ is well defined by $s^{'}(\psi(x))=\psi(x)$ and $s^{'}(i(x))=i(x)$ but in $D$ we have $i(x)=\psi(x)$ for all $x\in X$. We resume the situation in the following diagram




We construct the homotopy $H^{'}:D\times[0,1]\to D$ as follow:

1. $H^{'}(-,0)=id_D$.
2. $H^{'}(z,t)=z$ if $z$ is in $Z$ .
3. $H^{'}(y,t)=H(y,t)$ for all $y$ in $Y$.

This homotopy is well defined. In fact, it is enough to prove that the gluing operation is well defined. We have $\psi(x)=i(x)$ in $D$, then $H^{'}(i(x),t)=H(i(x),t)=i(x)$ by definition, on the other hand $H^{'}(\psi(x),t)=\psi(x)$. Since $i(X)$ is closed in

$Y$, then $i(X)$ is closed in $D$. We conclude that $H^{'}$ is well defined. Moreover, $H^{'}(y,0) = H(y,0) = y$ and so $H^{'}(-,0)$ is the identity.

By simple computation of $H^{'}(-,1) : D \to D$ we have that $H^{'}(z,1) = z$ for all $z \in Z$ and $H^{'}(y,1) = H(y,1) = s(y)$ for all $y \in Y$. So, $H^{'}(-,1) = s'$. That means the morphism $s^{'} : D \to Z \subset D$ is a weak equivalence since it is homotopic to the identity. Consequently $g$ is also a homotopy equivalence because $s^{'} \circ g = id$.

□

**Lemma 4.4.** *With the precedent notation of section 4, if we pose* $f: \ S = |\Lambda^n_i| \to T = |\Delta^n|$, *then,* $\mathbf{Map}_{\mathbf{C}}(w,z) \subset \mathbf{Map}_{\mathbf{D}}(w,z)$ *is a weak equivalence* $\forall w, \ z \in \mathbf{C}$.

*Proof.* We remind here that $\mathbf{V} = \mathbf{Top}$. Since all objects in **Top** are fibrant, $f$ admits a section $s$. On the other hand, the inclusion $N^{k+1}_{\mathbf{C}} \subset M^{k+1}_{\mathbf{C}}$ is a weak equivalence and admits also a section. We will do the demonstration for the case $k = 2$. We use the following notations:

$$A_0 = \mathbf{Map}_{\mathbf{C}}(y,z) \times S \times \mathbf{Map}_{\mathbf{C}}(y,x) \times S \times \mathbf{Map}_{\mathbf{C}}(w,x) \tag{4.1}$$

$$A_1 = \mathbf{Map}_{\mathbf{C}}(y,z) \times S \times \mathbf{Map}_{\mathbf{C}}(y,x) \times T \times \mathbf{Map}_{\mathbf{C}}(w,x) \tag{4.2}$$

$$A_2 = \mathbf{Map}_{\mathbf{C}}(y,z) \times T \times \mathbf{Map}_{\mathbf{C}}(y,x) \times S \times \mathbf{Map}_{\mathbf{C}}(w,x). \tag{4.3}$$

The evident inclusions are weak equivalences which admit sections induced by $s$ $A_0 \to A_i, \ i = 1,2$.

We define the complement of $N^2_{\mathbf{C}}$, which consists on tuples $(a, s_1, b, s_2, c)$ in $\mathbf{Map}_{\mathbf{C}}(y,z) \times T \times \mathbf{Map}_{\mathbf{C}}(y,x) \times T \times \mathbf{Map}_{\mathbf{C}}(w,x)$ such that $s_1, \ s_2 \notin S$. We will do our argument in low dimension $n = 1$, the rest is similar. The space $T \times S \cup_{S\times S} S \times T$ is a gluing of two intervals $[0,1]$ at the point 0 and $T \times T$ is simply $[0,1] \times [0,1]$. If we pose $f: \ X = T \times S \cup_{S\times S} S \times T \to T \times T = Y$, we are exactly in the situation of the lemma 4.3 i.e., there exists a homotopy between $X$ and $Y$ which is identity map on $X$. If we rewrite $N^2_{\mathbf{C}}$ by

$$N^2_{\mathbf{C}} = A_1 \bigcup_{A_0} A_2 = X \times \mathbf{Map}_{\mathbf{C}}(y,z) \times \mathbf{Map}_{\mathbf{C}}(y,x) \times \mathbf{Map}_{\mathbf{C}}(w,x),$$

and $M^2_{\mathbf{C}}$ by

$$M^2_{\mathbf{C}} = Y \times \mathbf{Map}_{\mathbf{C}}(y,z) \times \mathbf{Map}_{\mathbf{C}}(y,x) \times \mathbf{Map}_{\mathbf{C}}(w,x),$$

The induced morphism $N^2_{\mathbf{C}} \to M^2_{\mathbf{C}}$ verify the condition of the lemma 4.3. Consequently, the pushout of $N^2_{\mathbf{C}} \subset M^2_{\mathbf{C}}$ along $N^2_{\mathbf{C}} \to \mathbf{Map}_D(w,z)^1$ is also a weak equivalence. It means that the inclusion $\mathbf{Map}_D(w,z)^1 \subset \mathbf{Map}_D(w,z)^2$ is a weak equivalence. By the same argument we prove the statement for all $k$ and use the fact that a transfinite composition of weak equivalences is a weak equivalence. So

$$\mathbf{Map}_{\mathbf{C}}(w,z) \cdots \subset \mathbf{Map}_{\mathbf{D}}(w,z)^k \subset \mathbf{Map}_{\mathbf{D}}(w,z)^{k+1} \cdots \subset \mathbf{Map}_{\mathbf{D}}(w,z)$$

is a weak equivalence.

□

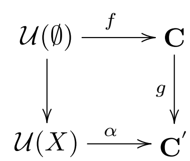
**Lemma 4.5.** *Let* $\mathbf{C} \to \mathbf{D}$ *a weak equivalence in* $\mathcal{O} - \mathbf{Cat}_{\mathbf{Top}}$ *and let* $\mathbf{\Gamma}$ *a graph in* $\mathcal{O} - \mathbf{Graph}_{\mathbf{Top}}$, *the induced morphism :*

$$L(\mathbf{\Gamma}) \star \mathbf{C} \to L(\mathbf{\Gamma}) \star \mathbf{D}$$

*is a weak equivalence in the category* $\mathcal{O} - \mathbf{Cat}_{\mathbf{Top}}$.

*Proof.* It is enough to prove that $\mathbf{C}^{'} = L(\mathbf{\Gamma})_{a,b} \star \mathbf{C} \to L(\mathbf{\Gamma})_{a,b} \star \mathbf{D} = \mathbf{D}^{'}$ is an equivalence for all $(a, b) \in \mathcal{O} \times \mathcal{O}$. If $a \neq b$, it is a direct consequence of the lemma 4.4, where we replace $S$ by $\emptyset$ and $T$ by $X$. So $\mathbf{Map}_{\mathbf{C}'}(w, z) = \bigsqcup_k M^k_{\mathbf{C}}$ and respectively $\mathbf{Map}_{\mathbf{D}'}(w, z) = \bigsqcup_k M^k_{\mathbf{D}}$. But $M^k_{\mathbf{C}}$ is equivalent to $M^k_{\mathbf{D}}$ since $\mathbf{C}$ is equivalent to $\mathbf{D}$. We conclude that $\mathbf{Map}_{\mathbf{C}'}(w, z)$ is equivalent to $\mathbf{Map}_{\mathbf{D}'}(w, z)$.

If $a = b$, we note the edges from $a$ to $a$ of the graph $\mathbf{\Gamma}$ by $X$. Then we use the precedent case if we remark that $\mathbf{C}^{'} = L(\mathbf{\Gamma})_{a,b} \star \mathbf{C}$ is simply the following pushout:




The morphism $f$ send the two objects of $\mathcal{U}(\emptyset)$ to $a \in \mathbf{C}$, so, by the lemma 4.4 we have that $L(\mathbf{\Gamma})_{a,a} \star \mathbf{C} \to L(\mathbf{\Gamma})_{a,a} \star \mathbf{D}$ is a weak equivalence. Consequently $L(\mathbf{\Gamma}) \star \mathbf{C} \to L(\mathbf{\Gamma}) \star \mathbf{D}$ is a weak equivalence by a possibly transfinite composition of weak equivalences. □

**Corollary 4.6.** *Let* $\mathbf{M}$ *in* $\mathcal{O} - \mathbf{Cat_{Top}}$*, then* $F_i\mathbf{M} \star \mathbf{C} \to F_i\mathbf{M} \star \mathbf{D}$ *is a weak equivalence in* $\mathcal{O} - \mathbf{Cat_{Top}}$ *for all* $0 \leq i$.

*Proof.* It is enough to see that $F = LU$ and applied the lemma 4.5 by putting $\mathbf{\Gamma} = U\mathbf{M}$. □

**Lemma 4.7.** *Let* $\mathbf{C}$, $\mathbf{D}$ *and* $\mathbf{M}$ *in* $\mathcal{O} - \mathbf{Cat_{Top}}$ *, and* $\mathbf{C} \to \mathbf{D}$ *a weak equivalence. Then*

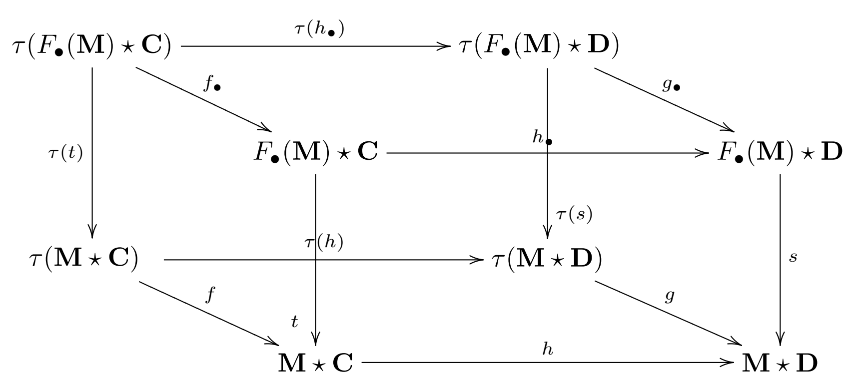
$$\mathbf{M} \star \mathbf{C} \to \mathbf{M} \star \mathbf{D}$$

*is a weak equivalence.*

*Proof.* We have seen by 4.6 that

$$h_i : F_i(\mathbf{M}) \star \mathbf{C} \to F_i(\mathbf{M}) \star \mathbf{D}$$

is a weak equivalence for all $0 \leq i$. Consider the following commutative diagram in $\mathcal{O} - \mathbf{Graph_{sTop}}$:




The morphism $t$ and $s$ are homotopy equivalences. By 3.4, the morphisms $|t|$ and $|s|$ are also homotopy equivalences (of underling graphs).
The morphisms $\tau(t)$ and $\tau(s)$ are homotopy equivalences by 3.11. And by 3.4, the morphisms $|\tau(t)|$ and $|\tau(s)|$ are homotopy equivalences.

The morphism $|\tau(h_\bullet)|$ is a weak equivalence by 3.7.
By the property *2 out of 3*, $|\tau(h)|$ is a weak equivalence.
The morphisms $f$ and $g$ are homotopy equivalences by 3.12. So $|f|$ and $|g|$ are also homotopy equivalences by 3.4.
We conclude by the property "2 out of 3" that $|h|$ is a weak equivalence and so $h$ is a weak equivalence.

□

## 5. ∞-CATEGORIES (QUASI-CATEGORIES)

In the mathematical literature, there are many models for ∞-categories, for example the enriched categories on Kan complexes [2], The categories enriched over **Top** as we saw before, and the quasi-categories defined by Joyal. More precisely Joyal constructed a new model structure on **sSet**, see [7], where the fibrant object are by definition *quasi-categories* (i.e., ∞-categories). We introduce the notion of *quasi-groupoid* (i.e., ∞-groupoids) which generalize the notion of groupoids in the classical setting of categories. We remind also the definition of **coherent nerve** for enriched categories on **sSet** and **Top**.

**Definition 5.1.** A *quasi-category* is a simplicial set $X$ which has a lifting property for all $0 < i < n$:

$$\begin{array}{ccc} \Lambda^n_i & \xrightarrow{\forall} & X \\ \downarrow & \overset{\exists}{\nearrow} & \downarrow \\ \Delta^n & \longrightarrow & * \end{array} \tag{5.1}$$

It is important to remark that the condition $0 < i < n$ codify the composition law up to homotopy. Sometimes, we will call such simplicial complexes by weak Kan complexes. For example, if $\mathbf{C}$ is a classical category, then the nerve $\mathrm{N}_\bullet\mathbf{C}$ is a quasi-category with an additional property: the lifting, is in fact, unique (cf [9], proposition 1.1.2.2). Moreover, a simplicial set is isomorphic to the nerve of a category $\mathbf{C}$ if and only if the lifting 5.1 exists and is unique.

**Lemma 5.2.** *A category* $\mathbf{C}$ *is a groupoid iff* $\mathrm{N}_\bullet\mathbf{C}$ *is a Kan complex.*

*Proof.* If $\mathbf{C}$ is a groupoid, then $\mathrm{N}_\bullet\mathbf{C}$ admits a lifting with respect to $\Lambda^n_n \to \Delta^n$ and $\Lambda^n_0 \to \Delta^n$ simply because all arrows in $\mathbf{C}$ are invertible. So $\mathrm{N}_\bullet\mathbf{C}$ is a Kan complex. If $\mathrm{N}_\bullet\mathbf{C}$ is a Kan complex, we have a lifting with respect to $\Lambda^2_2 \to \Delta^2$. That means, every diagram in $\mathbf{C}$

$$\begin{array}{ccc} & & x \\ & \overset{f}{\nearrow} & \downarrow{\scriptstyle id} \\ y & \xrightarrow{g} & x \end{array}$$

can be completed by a unique arrow $f : y \to x$, so $g$ is right invertible. We show that $g$ is left invertible using the lifting property with respect to $\Lambda^2_0 \to \Delta^2$. So $\mathbf{C}$ is a groupoid. □

The precedent lemma suggest us a definition for an ∞-groupoid.

**Definition 5.3.** An ∞-category (quasi-category) $X$ is an ∞-groupoid (quasi-groupoid) if it is a Kan complex.

**Example 5.4.** Let $Y$ be a topological space, the simplicial set sing$Y$ is a Kan complex. so we can see every topological space as an $\infty$-groupoid.

**Theorem 5.5.** [7] *(section 6.3) The category* **sSet** *admits a model structure where the cofibrations are the monomorphisms, the fibrant objects are the quasi-categories, the fibrations are the pseudo-fibrations and the weak equivalences are the categorical equivalences. This is a cartesian closed model structure. This new structure is noted by* $(\mathbf{sSet}, \mathbf{Q})$.

We will explain later what we mean by categorical equivalences, but we don't describe explicitly the pseudo-fibration. For each quasi-category $X$ (fibrant object in $(\mathbf{sSet}, \mathbf{Q})$, we can associate its homotopy category (in a classical sense) noted Ho$X$. This theory was developed by Joyal, see for example [7].

## 6. Some Quillen adjunctions

**Notation 6.1.** We will note the category of simplicial sets with Kan model structure by $(\mathbf{sSet}, \mathbf{K})$. The Joyal model structure of quasi-categories will be noted by $(\mathbf{sSet}, \mathbf{Q})$.

In this paragraph, we describe different Quillen adjunction between $\mathbf{Cat_{sSet}}$, $(\mathbf{sSet}, \mathbf{Q})$ and $(\mathbf{sSet}, \mathbf{K})$.

### 6.1. $\mathbf{Cat_{sSet}}$ vs $(\mathbf{sSet}, \mathbf{Q})$.

The first adjunction is described in details in [9]. We start by some analogies between classical categories ann simplicial sets.

$$\mathbf{sSet} \underset{\mathrm{N}_\bullet}{\overset{\tau}{\rightleftarrows}} \mathbf{Cat},$$

the right adjoint is the nerve and the left adjoint associate to each simplicial set its fundamental category. Note that this adjunction is not a Quillen adjunction for the two known model structure on **Cat** ( Thomason structure and Joyal structure). We remind that the nerve functor is fully faithful and $\tau \mathrm{N}_\bullet = id$. The basic idea is to "extend" this adjunction to an adjunction between $(\mathbf{sSet}, \mathbf{Q})$ and the category $\mathbf{Cat_{sSet}}$. If we use the standard nerve for the enriched categories on simplicial sets, by remembering only the 0-simplices, then we will loose all the higher homotopical information. Because of that, we use an other strategy. First we define a left adjoint as follow

$$\Xi : (\mathbf{sSet}, \mathbf{Q}) \to \mathbf{Cat_{sSet}}$$

on $\Delta^n$, then we apply the left Kan extension.

**Definition 6.2.** [9] (1.1.5.1) The enriched category $\Xi(\Delta^n)$ has as objects the 0-simplices of $\Delta^n$, and

$$\Xi(\Delta^n)(i,j) = \begin{cases} \widetilde{\mathrm{N}_\bullet} P_{i,j} & \text{if } i \leq j \\ \emptyset & \text{if } i > j \end{cases}$$

Where $P_{i,j}$ is the set partially ordered by inclusion:

$$\{I \subseteq J : (i, j \in I) \wedge (\forall k \in I)[i \leq k \leq j]\}.$$

**Definition 6.3.** The right adjoint to the functor $\Xi$ is called the coherent nerve and noted by $\widetilde{\mathrm{N}_\bullet}$. It is defined by the following formula:

$$\widetilde{\mathrm{N}_n}\mathbf{C} = \mathbf{hom_{sSet}}(\Delta^n, \widetilde{\mathrm{N}_\bullet}\mathbf{C}) := \mathbf{hom_{Cat_{sSet}}}(\Xi(\Delta^n), \mathbf{C}).$$

Now, we can define the categorical equivalences used in the model structure $(\mathbf{sSet}, \mathbf{Q})$. We call a morphism of simplicial sets $f: X \to Y$ a categorical equivalence if $\Xi(f): \Xi(X) \to \Xi(Y)$ is an equivalence of enriched categories, i.e., if $\mathbf{Map}_{\Xi(X)}(a,b) \to \mathbf{Map}_{\Xi(Y)}(\Xi(f)a, \Xi(f)b)$ is a weak equivalence of simplicial sets for all $a,\ b$ and $\pi_0\Xi(f): \pi_0\Xi(X) \to \pi_0\Xi(Y)$ is a equivalence of classical categories.

**Theorem 6.4.** *The following adjunction is a Quillen equivalence between the Joyal model structure* $(\mathbf{sSet}, \mathbf{Q})$ [7], *and the model category on* $\mathbf{Cat_{sSet}}$ *defined in* [2]

$$\mathbf{sSet} \underset{\widetilde{\mathrm{N}_\bullet}}{\overset{\Xi}{\rightleftarrows}} \mathbf{Cat_{sSet}}.$$

For the proof we refer to [9] theorem 2.2.5.1.

**Corollary 6.5.** *Let* $\mathbf{C}$ *an enriched category on Kan complexes, then the counity*

$$\Xi\widetilde{\mathrm{N}_\bullet}\mathbf{C} \to \mathbf{C}$$

*is a weak equivalence of enriched categories.*

6.2. $(\mathbf{sSet}, \mathbf{Q})$ **vs** $(\mathbf{sSet}, \mathbf{K})$**.** In this paragraph, we describe the Quillen adjunction between Joyal model structure on simplicial sets and the classical model structure on **sSet** which we note by $(\mathbf{sSet}, \mathbf{K})$, $\mathbf{K}$ for Kan complexes.

**Definition 6.6.** The functor $k: \Delta \to \mathbf{sSet}$ is defined by $k([n]) = \mathrm{N}_\bullet[n]^{'}$ for all $n \geq 0$, where $\mathrm{N}_\bullet[n]^{'}$ is the nerve of the free groupoid generated by the category $[n]$. If $X$ is a simplicial set, we define the functor $k^! : \mathbf{sSet} \to \mathbf{sSet}$ by :

$$k^!(X)_n = \mathbf{hom_{sSet}}(\mathrm{N}_\bullet[n]^{'}, X).$$

The functor $k^!$ has a left adjoint $k_!$ which is the left Kan extension of $k$. From the inclusion $\Delta^n \subset \widetilde{\Delta^n}$ we obtain, for all $n$, a set morphism $k^!(X)_n \to X_n$ which is $n$-level of a simplicial morphism $\beta_X: k^!(X) \to X$. More precisely, $\beta: k^! \to id$ is a natural transformation. Dually, we define a natural transformation $\alpha: id \to k_!$

**Theorem 6.7.** *The adjoint functors*

$$(\mathbf{sSet}, \mathbf{Kan}) \underset{k^!}{\overset{k_!}{\rightleftarrows}} (\mathbf{sSet}, \mathbf{Q}).$$

*is a Quillen adjunction. Moreover,* $\alpha_X: X \to k_!(X)$ *is an equivalence for each* $X$.

*Proof.* For the proof, see ([7], 6.22). □

6.3. $\infty$**-groupoids.** In this paragraph, we define a notion of groupoid for categories enriched on simplicial sets or topological spaces, Which we compare with the notion of $\infty$-groupoid defined for quasi-categories.

**Definition 6.8.** An enriched category $\mathbf{C}$ on **sSet** (or **Top**) is an $\infty$-groupoid if $\pi_0\mathbf{C}$ is a groupoid in the classical sense of categories. If $\mathbf{C}$ is enriched on **sSet** (**Top**), the $\infty$-groupoid $G\mathbf{C}$ associated to $\mathbf{C}$ is a fibered product in $\mathbf{Cat_{sSet}}$ (or $\mathbf{Cat_{Top}}$):

$$\begin{array}{ccc} G\mathbf{C} = \mathrm{iso}\pi_0\mathbf{C} \times_{\pi_0\mathbf{C}} \mathbf{C} & \longrightarrow & \mathbf{C} \\ \downarrow & & \downarrow \\ \mathrm{iso}\pi_0\mathbf{C} & \longrightarrow & \pi_0\mathbf{C}. \end{array}$$

We remark that the functor $\pi_0 : \ \mathbf{Cat_{sSet}} \to \mathbf{Cat}$ is a left adjoint, so it does not necessary commute with limits in general. But the evident projection $pr$ : $\pi_0 G\mathbf{C} \to \mathrm{iso}\pi_0\mathbf{C}$ is an isomorphism. In fact, if $w_1$ and $w_2$ are weak equivalences in $\mathbf{Map_C}(a,b)$ and $h$ is a homotopy between them (i.e. un 1-simplex in $\mathbf{Map_C}(a,b)$ such that the borders are $w_1,\ w_2$), then $h$ is also a homotopy in $\mathbf{Map}_{G\mathbf{C}}(a,b)$. This prove that the projection $pr$ is fully faithful. the essential surjectivity of $pr$ is evident.
The full subcategory of $\mathbf{Cat_{sSet}}$ of $\infty$-groupoids is noted by $\mathbf{Grp_{sSet}}$.

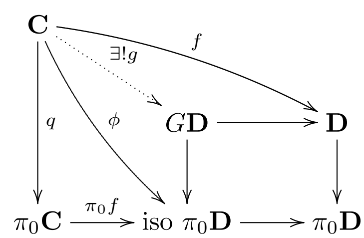
**Lemma 6.9.** *The functor $G$ : $\mathbf{Cat_{sSet}} \to \mathbf{Grp_{sSet}}$ is the right adjoint of the inclusion, i.e.,*

$$\mathbf{hom}_{\mathbf{Grp_{sSet}}}(\mathbf{C}, G\mathbf{D}) = \mathbf{hom}_{\mathbf{Cat_{sSet}}}(\mathbf{C}, \mathbf{D})$$

$\forall \mathbf{C} \in \mathbf{Grp_{sSet}}$ *and* $\mathbf{D} \in \mathbf{Cat_{sSet}}$.

**Remark 6.10.** We can do the same thing for $\mathbf{Cat_{Top}}$.

*Proof.* Let $\mathbf{C}$ be an $\infty$-groupoid and let $\mathbf{D} \in \mathbf{Cat_{sSet}}$. A morphism $f : \ \mathbf{C} \to \mathbf{D}$ define in a unique way an adjoint morphism $g : \ \mathbf{C} \to G\mathbf{D}$ given by the universal map




The morphism $\phi = \pi_0 f \circ q$ exists and make the diagram commuting, since $\mathbf{C}$ is an $\infty$-groupoid.

□

Let $[n]^{'}$ denote the groupoid freely generated by the category $[n]$. An example of $\infty$-groupoid is the category $\Xi k_! \Delta^n$. In fact, $\Xi k_! \Delta^n = \Xi \widetilde{\mathrm{N}_\bullet}[n]^{'} \to [n]^{'}$ is a weak categorical equivalence and $[n]^{'}$ is fibrant. Since $[n]^{'}$ is a groupoid, then $\pi_0 \Xi k_! \Delta^n$ is also a groupoid .

**Lemma 6.11.** *Let $\mathbf{C}$ a fibrant category enriched on* $\mathbf{sSet}$*, then $k^! \widetilde{\mathrm{N}_\bullet}\mathbf{C} = k^! \widetilde{\mathrm{N}_\bullet} G\mathbf{C}$, where $G\mathbf{C}$ is an $\infty$-groupoid associated to $\mathbf{C}$.*

*Proof.* Using the precedent adjunctions, we have for all $n \geq 0$

$$(k^! \widetilde{\mathrm{N}_\bullet}\mathbf{C})_n = \mathbf{hom_{sSet}}(\Delta^n, k^! \widetilde{\mathrm{N}_\bullet}\mathbf{C}) \tag{6.1}$$

$$= \mathbf{hom_{sSet}}(k_! \Delta^n, \widetilde{\mathrm{N}_\bullet}\mathbf{C}) \tag{6.2}$$

$$= \mathbf{hom}_{\mathbf{Cat_{sSet}}}(\Xi k_! \Delta^n, \mathbf{C}) \tag{6.3}$$

But $\Xi k_! \Delta^n$ is an $\infty$-groupoid, so

$$\mathbf{hom}_{\mathbf{Cat_{sSet}}}(\Xi k_! \Delta^n, \mathbf{C}) = \mathbf{hom}_{\mathbf{Grp_{sSet}}}(\Xi k_! \Delta^n, G\mathbf{C}) \tag{6.4}$$

$$= \mathbf{hom}_{\mathbf{Cat_{sSet}}}(\Xi k_! \Delta^n, G\mathbf{C}) \tag{6.5}$$

$$= \mathbf{hom_{sSet}}(\Delta^n, k^! \widetilde{\mathrm{N}_\bullet} G\mathbf{C}) \tag{6.6}$$

$$= (k^! \widetilde{\mathrm{N}_\bullet} G\mathbf{C})_n \tag{6.7}$$

we conclude that $k^!\widetilde{\mathrm{N}_\bullet} G\mathbf{C} = k^!\widetilde{\mathrm{N}_\bullet}\mathbf{C}$. □

**Definition 6.12.** [2] In Bergner's model structure on $\mathbf{Cat_{sSet}}$ [2] a morphism $F : \mathbf{C} \to \mathbf{D}$ is a fibration if

(1) $\mathbf{Map_C}(a, b) \to \mathbf{Map_D}(Fa, Fb)$ is a fibration of simplicial sets for all $a,\ b \in \mathbf{C}$.
(2) $F$ has a lifting property of weak equivalences, i.e. it is Grothendieck fibration for weak equivalences.

**Corollary 6.13.** *Let $G\mathbf{C}$ the $\infty$-groupoid associated to the enriched category $\mathbf{C}$ over Kan complexes (or $\mathbf{Top}$), then*

$$\widetilde{\mathrm{N}_\bullet} G\mathbf{C} \to \mathrm{N}_\bullet \mathrm{iso}\ \pi_0\mathbf{C}$$

*pseudo-fibration (cf.* [7] *) in* $(\mathbf{sSet}, \mathbf{Q})$*.*

*Proof.* Remark that if $\mathbf{C}$ is fibrant, then $\mathbf{C} \to \pi_0\mathbf{C}$ is a fibration. The Bergner's model structure is right proper so $G\mathbf{C} \to \mathrm{iso}\ \pi_0\mathbf{C}$ is also a fibration. Moreover, the groupoid $\mathrm{iso}\pi_0\mathbf{C}$ is fibrant, and so $G\mathbf{C}$ is. Consequently $\widetilde{\mathrm{N}_\bullet} G\mathbf{C} \to \widetilde{\mathrm{N}_\bullet}\mathrm{iso}\ \pi_0\mathbf{C}$ is a pseudo-fibration in the category $(\mathbf{sSet}, \mathbf{Q})$, So a pseudo fibration between quasi-categories.

But the category $\pi_0\mathbf{C}$ is a *constant* simplicial category, so $\widetilde{\mathrm{N}_\bullet}\mathrm{iso}\ \pi_0\mathbf{C} = \mathrm{N}_\bullet\mathrm{iso}\ \pi_0\mathbf{C}$. We conclude that $\widetilde{\mathrm{N}_\bullet} G\mathbf{C} \to \mathrm{N}_\bullet\mathrm{iso}\ \pi_0\mathbf{C}$ is a peudo-fibration between quasi-category and a Kan complex, see 5.2. □

Let $X$ a quasi-category, Joyal defined the homotopy category $\mathrm{Ho}(X)$ which is a category in the classical sense. The 0-simplexes of $X$ form the set of objets of $\mathrm{Ho}(X)$ and the 1-simplexes (modulo the homotopy equivalence) form the morphisms of $\mathrm{Ho}(X)$. A 1-simplex in $X$ is called an weak equivalence if it is represented in $\mathrm{Ho}(X)$ by an isomorphism.

**Definition 6.14.** Let $p : X \to Y$ a morphism between quasi-categories, and let $w$ a 1-simplex in $X$, then $p$ is called conservative if:

$$p(w) \text{ a weak equivalence in Y } \Rightarrow\ w \text{ a weak equivalence in X.}$$

**Lemma 6.15.** *(*[7]*, 4.30) Let $p : X \to Y$ be a morphism between quasi-categories, such that $p$ is a pseudo-fibration and conservative. If $Y$ is a Kan complex, then $X$ is.*

**Lemma 6.16.** *Let $\mathbf{C} \in \mathbf{Cat_{sSet}}$ be fibrant object, then $\widetilde{\mathrm{N}_\bullet} G\mathbf{C}$ is a Kan complex, where $G\mathbf{C}$ is the $\infty$-groupoid associated to $\mathbf{C}$.*

*Proof.* We have seen by the corollary 6.13 that if $\mathbf{C}$ is fibrant, then $\widetilde{\mathrm{N}_\bullet} G\mathbf{C} \to \mathrm{N}_\bullet\mathrm{iso}\ \pi_0\mathbf{C}$ is a pseudo-fibration between quasi-categories, and $\mathrm{N}_\bullet\mathrm{iso}\ \pi_0\mathbf{C}$ is a Kan complex. We must verify that the morphism is conservative, which is an evident fact because all 0-simplices of $\mathbf{Map}_{G\mathbf{C}}(a, b)$ are weak equivalences by definition. By the lemma 6.15, we conclude that $\widetilde{\mathrm{N}_\bullet} G\mathbf{C}$ is a Kan complex. □

In [7] (Theorem 4.19) , Joyal construct an adjunction between Kan complexes and quasi-categories. If we note by **Kan** the full subcategory of **sSet** of Kan complexes, and by **QCat** the full subcategory of **sSet** of quasi-categories, then the inclusion $\mathbf{Kan} \subset \mathbf{QCat}$ admits a right adjoint noted by J. The functor can be interpreted as follow: for each quasi-category $X$, $\mathrm{J}(X)$ is the quasi-groupoid associated to $X$, and if $X$ is a Kan complex, then $\mathrm{J}(X) = X$.

**Lemma 6.17.** *Let $X$ be a quasi-category (a fibrant object) in $(\mathbf{sSet}, \mathbf{Q})$. The natural transformation $\beta_X : k^!(X) \to X$ is factored by $\beta_X : k^!(X) \to \mathrm{J}(X) \subset X$. Moreover, $\beta_X : k^!(X) \to \mathrm{J}(X)$ is a trivial Kan fibration.*

*Proof.* See [7], proposition 6.26. □

**Corollary 6.18.** *Let $\mathbf{C} \in \mathbf{Cat_{sSet}}$ be a fibrant category , and $G\mathbf{C}$ the associated $\infty$-groupoid. Then $k^!\widetilde{\mathrm{N}_\bullet}(\mathbf{C}) \to \widetilde{\mathrm{N}_\bullet}(G\mathbf{C})$ is a trivial Kan fibration.*

*Proof.* Since $\mathbf{C}$ is fibrant, we have seen that $k^!\widetilde{\mathrm{N}_\bullet}(\mathbf{C}) = k^!\widetilde{\mathrm{N}_\bullet}(G\mathbf{C})$, and by the precedent lemma $k^!\widetilde{\mathrm{N}_\bullet}(G\mathbf{C}) \to \mathrm{J}(\widetilde{\mathrm{N}_\bullet}(G\mathbf{C}))$ is a trivial Kan fibration. But $\widetilde{\mathrm{N}_\bullet}(G\mathbf{C})$ is a Kan complex, since $G\mathbf{C}$ is a fibrant $\infty$-groupoid, so $\mathrm{J}(\widetilde{\mathrm{N}_\bullet}(G\mathbf{C})) = \widetilde{\mathrm{N}_\bullet}(G\mathbf{C})$. □

Now, we can see the analogy between $\mathrm{N}_\bullet\mathrm{iso}$ in the case of classical categories and the functor $k^!\widetilde{\mathrm{N}_\bullet}$ in the case of enriched categories over **sSet**. In fact, if $\mathbf{C}$ is a classical category, then the functor iso sends $\mathbf{C}$ to its associated groupoid $G\mathbf{C}$ and so $\mathrm{N}_\bullet\mathrm{iso}\mathbf{C} = \mathrm{N}_\bullet G\mathbf{C}$. If $\mathbf{C}$ is a category enriched over Kan complexes,( i.e., $\mathbf{C}$ is fibrant in Bergner's model structure), then the simplicial set $k^!\widetilde{\mathrm{N}_\bullet}\mathbf{C}$ is equivalent to $\widetilde{\mathrm{N}_\bullet}G\mathbf{C}$ by the corollary 6.18.

## 7. MAPPING SPACE

The goal of this section is to describe the mapping space of the model category $\mathbf{Cat_{Top}}$.

**Theorem 7.1.** *[[3], theorem 2.12.] Let a Quillen adjunction of Quillen model categories :*

$$\mathbf{C} \underset{Z}{\overset{W}{\rightleftarrows}} \mathbf{D}.$$

*The there is a natural isomorphism*

$$\mathbf{map_C}(a, \mathbb{R}Zb) \to \mathbf{map_D}(\mathbb{L}Wa, b)$$

*in* $\mathrm{Ho}(\mathbf{sSet})$

7.1. **Mapping space in $\mathbf{Cat_{Top}}$ and $\mathbf{Cat_{sSet}}$.** Suppose that $\mathbf{C}$ is a small enriched category on **Top**. We define the coherent nerve of $\mathbf{C}$ by $\widetilde{\mathrm{N}_\bullet}\mathrm{sing}\mathbf{C}$, and we define the corresponding $\infty$- groupoid $G\mathbf{C}$ by

$$\begin{array}{ccc} G\mathbf{C} = \mathrm{iso}\ \pi_0\mathbf{C} \times_{\pi_0\mathbf{C}} \mathbf{C} & \longrightarrow & \mathbf{C} \\ \downarrow & & \downarrow \\ \mathrm{iso}\ \pi_0\mathbf{C} & \longrightarrow & \pi_0\mathbf{C} \end{array}$$

By applying the functor sing to this diagram, we obtain also a pullback diagram since sing since it is a right adjoint. We note that $\mathrm{sing}\ \pi_0\mathbf{C} = \pi_0\mathrm{sing}\ \mathbf{C} = \pi_0\mathbf{C}$ and $\mathrm{sing}\ \mathrm{iso}\pi_0\mathbf{C} = \mathrm{iso}\ \pi_0\mathbf{C} = \mathrm{iso}\ \pi_0\mathrm{sing}\mathbf{C}$

$$\begin{array}{ccc} G\ \mathrm{sing}\ \mathbf{C} = \mathrm{sing}(\mathrm{iso}\pi_0\mathbf{C} \times_{\pi_0\mathbf{C}} \mathbf{C}) & \longrightarrow & \mathrm{sing}\ \mathbf{C} \\ \downarrow & & \downarrow \\ \mathrm{sing}\ \mathrm{iso}\ \pi_0\mathbf{C} & \longrightarrow & \mathrm{sing}\ \pi_0\mathbf{C} \end{array}$$

We conclude that

$$\mathrm{sing}\ G\mathbf{C} = G\ \mathrm{sing}\ \mathbf{C}.$$

Moreover, $k^! \ \widetilde{\mathrm{N}_\bullet}$ sing $\mathbf{C}$ is weak equivalent to $\widetilde{\mathrm{N}_\bullet}$ sing $G\mathbf{C}$.

**Corollary 7.2.** *For every (fibrant) small category enriched on* $\mathbf{sSet}$*, we have the following isomorphism in* $\mathrm{Ho}(\mathbf{sSet})$

$$k^!\widetilde{\mathrm{N}_\bullet}\mathbf{C} \sim \mathbf{map_{sSet}}(*, k^!\widetilde{\mathrm{N}_\bullet}\mathbf{C}) \sim \widetilde{\mathrm{N}_\bullet}G\mathbf{C} \sim \mathbf{map_{Cat_{sSet}}}(*, \mathbf{C})$$

*and by the same way, if* $\mathbf{D}$ *is a small category enriched on* $\mathbf{Top}$*, then*

$$\mathbf{map_{Cat_{Top}}}(*, \mathbf{D}) \sim k^!\widetilde{\mathrm{N}_\bullet}\mathrm{sing}\mathbf{D} \sim \widetilde{\mathrm{N}_\bullet}G\mathrm{sing}\mathbf{D}.$$

*More generally, we have that:*

$$\mathbf{map_{Cat_{Top}}}(|\Xi k_!(A)|, \mathbf{D}) \sim \mathbf{map_{sSet}}(A, k^!\widetilde{\mathrm{N}_\bullet}\mathrm{sing}\mathbf{D}) \sim \mathbf{Map}(A, \widetilde{\mathrm{N}_\bullet}G\mathrm{sing}\mathbf{D}),$$

*where* $\mathbf{Map}$ *is the right adjoint functor to the cartesian product in* $\mathbf{sSet}$ *and* $A$ *any simplicial set.*

*Proof.* The homotopy type of the mapping space $\mathbf{map_{Cat_{Top}}}(*, \mathbf{C})$ is computed easily using the theorem 7.1, and the adjunction

$$\mathbf{sSet} \underset{k^!\widetilde{\mathrm{N}_\bullet}}{\overset{\Xi k_!}{\rightleftarrows}} \mathbf{Cat_{sSet}}.$$

and by the corollary 6.18, we conclude that

$$\mathbf{map_{Cat_{sSet}}}(*, \mathbf{C}) \sim \widetilde{\mathrm{N}_\bullet}G\mathbf{C}.$$

respectively

$$\mathbf{map_{Cat_{Top}}}(*, \mathbf{D}) \sim \widetilde{\mathrm{N}_\bullet}G\mathrm{sing}\mathbf{D}.$$

□

In the classical setting of $\mathbf{Cat}$, we know that $\mathbf{map_{Cat}}(\mathbf{A}, \mathbf{B}) \sim \mathrm{N}_\bullet\mathrm{iso}\mathbf{HOM_{Cat}}(\mathbf{A}, \mathbf{B})$. If $\mathbf{A}$ is the equivalent to the terminal category $*$, then $\mathbf{map_{Cat}}(*, \mathbf{B}) \sim \mathrm{N}_\bullet\mathrm{iso}\mathbf{B}$.

Now, the similarity between $\mathbf{Cat}$ and $\mathbf{Cat_{sSet}}$ ($\mathbf{Cat_{Top}}$) is evident.

Following the article [11] in which there is an explicit description of the mapping space the category of $dg$-categories denoted by $dg-\mathbf{Cat}$. B. Toën proved that the category $\mathrm{Ho}(dg-\mathbf{Cat})$ is symmetric monoidal **closed** category and so :

$$\mathbf{map}_{dg-\mathbf{Cat}}(\mathbf{M} \otimes^{\mathbb{L}} \mathbf{C}, \mathbf{D}) \sim \mathbf{map}_{dg-\mathbf{Cat}}(\mathbf{M}, \mathbb{R}\mathbf{HOM}(\mathbf{C}, \mathbf{D})).$$

Essentially, we can adopt the same prove to show that $\mathrm{Ho}\mathbf{Cat_V}$ is symmetric monoidal **closed** category.

Moreover, $\mathbb{R}\mathbf{HOM}(\mathbf{C}, \mathbf{D})$ is the $\mathbf{V}$-category of right quasi representable (fibrant-cofibrant) [11] $\mathbf{C}^{op} \times^{\mathbb{L}} \mathbf{D}$-modules.

**Corollary 7.3.** *Let* $\mathbf{D}$ *be a topological category or fibrant simplicial category and* $\mathbf{C}$ *a cofibrant object in* $\mathbf{Cat_V}$ *then:*

$$\mathbf{map_{Cat_V}}(\mathbf{C}, \mathbf{D}) \sim \mathrm{N}_\bullet\ w\mathbb{R}\mathbf{HOM}(\mathbf{C}, \mathbf{D}) \sim \widetilde{\mathrm{N}_\bullet}G\mathbb{R}\mathbf{HOM}(\mathbf{C}, \mathbf{D})$$

*where* $G\mathbb{R}\mathbf{HOM}(\mathbf{C},\mathbf{D})$ *the* $\infty$*-groupoid associated to* $\mathbb{R}\mathbf{HOM}(\mathbf{C},\mathbf{D})$ *,* $w\mathbb{R}\mathbf{HOM}(\mathbf{C},\mathbf{D})$ *is the discrete subcategory of weak equivalences and* $\mathrm{N}_\bullet$ *the standard nerve of discrete categories (enriched over* **Set***). and more specially*

$$\widetilde{\mathrm{N}_\bullet} G\mathbf{D} \sim \mathrm{N}_\bullet w\ \mathbb{R}\mathbf{HOM}(*,\mathbf{D}) \sim \widetilde{\mathrm{N}_\bullet} G\mathbb{R}\mathbf{HOM}(*,\mathbf{D}).$$

*Proof.* By 7.1 we have that $\mathbf{map}_{\mathbf{Cat_V}}(\mathbf{C},\mathbf{D}) \sim \mathbf{map}_{\mathbf{Cat_V}}(*,\mathbb{R}\mathbf{HOM}(\mathbf{C},\mathbf{D}))$. By corollary 7.2 we have that $\mathbf{map}_{\mathbf{Cat_V}}(*,\mathbb{R}\mathbf{HOM}(\mathbf{C},\mathbf{D})) \sim \widetilde{\mathrm{N}_\bullet} G\mathbb{R}\mathbf{HOM}(\mathbf{C},\mathbf{D})$. Finally by [11], $\mathbf{map}_{\mathbf{Cat_V}}(\mathbf{C},\mathbf{D}) \sim \mathrm{N}_\bullet w\mathbb{R}\mathbf{HOM}(\mathbf{C},\mathbf{D})$. □

Now we state a surprising result, which relate the enriched case to the discrete one.

**Corollary 7.4.** *Suppose that* $\mathbf{D} = \mathbb{R}\mathbf{HOM}(*,\mathbf{C})$ *then*

$$\widetilde{\mathrm{N}_\bullet} G\mathbf{D} \sim \mathrm{N}_\bullet\ w\mathbf{D}$$

*Proof.* We have seen that $\mathrm{N}_\bullet G\mathbf{D} \sim \mathrm{N}_\bullet w\mathbb{R}\mathbf{HOM}(*,\mathbf{D})$. Replacing $\mathbf{D}$ by $\mathbb{R}\mathbf{HOM}(*,\mathbf{C})$ we obtain

$$\widetilde{\mathrm{N}_\bullet} G\mathbf{D} \sim \mathrm{N}_\bullet w\mathbb{R}\mathbf{HOM}(*,\mathbb{R}\mathbf{HOM}(*,\mathbf{C})) \sim \mathbf{map}_{\mathbf{Cat_V}}(* \times^{\mathbb{L}} *,\mathbf{C}) \sim \mathrm{N}_\bullet w\ \mathbf{D}.$$

□

## 8. LOCALIZATION

In this paragraph we show how to construct a localization for a topological category with respect to a morphism or a set of morphisms. In the classical setting of small categories we know how to define the localization in a functorial way. The idea is quite simple, let $\mathbf{C} \in \mathbf{Cat}$ and $f$ be a morphism in $\mathbf{C}$, we want to define a functor $\mathbf{C} \to \mathrm{L}_f\mathbf{C}$ and having the following universal property: if $F : \mathbf{C} \to \mathbf{D}$ is a functor such that $F(f)$ is an isomorphism in $\mathbf{D}$ then there is a unique factorization of $F$ as

$$\mathbf{C} \to \mathrm{L}_f\mathbf{C} \to \mathbf{D}.$$

**Notation 8.1.** In this section, the category with two objects $x$ and $y$ and with one non trivial morphism from $x$ to $y$ will be denoted $\mathbf{A}$.
The category with the same objects $x$ and $y$ and an isomorphism from $x$ to $y$ (resp. from $y$ to $x$ ) will be denoted $\mathbf{B}$.

**Lemma 8.2.** *The category* $\mathrm{L}_f\mathbf{C}$ *is isomorphic to following pushout in* **Cat***:*

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{f} & \mathbf{C} \\ {\scriptstyle inc}\downarrow & & \downarrow{\scriptstyle i} \\ \mathbf{B} & \longrightarrow & \mathbf{M} \end{array}$$

*Where inc is the evident inclusion and* $f$ *sends the unique arrow in* $\mathbf{A}$ *to the morphism* $f$ *in* $\mathbf{C}$.

*Proof.* Suppose that we have a functor $F : \mathbf{C} \to \mathbf{D}$ such that the morphism $f$ is sent to an isomorphism. It induce a functor from $\mathbf{B} \to \mathbf{D}$. By the pushout property we have a unique functor from $\mathbf{M}$ to $\mathbf{D}$ which factors the functor $F$. So $\mathrm{L}_f\mathbf{C}$ is isomorphic to $\mathbf{M}$. □

**Corollary 8.3.** *For any set $S$ of morphism in $\mathbf{C}$ the category $\mathrm{L}_S\mathbf{C}$ exist and it is unique up to isomorphism.*

Now, we are interested for the same construction in the enriched setting $\mathbf{Cat_{Top}}$. The main difference with the classical case is the the existence, we will construct a functorial model for the localization up to homotopy.

**Notation 8.4.** We denote by $\mathbf{A}^h$ the topological category $|\Xi \mathrm{N}_\bullet \mathbf{A}|$ and by $\mathbf{B}^h$ the category $|\Xi \mathrm{N}_\bullet \mathbf{B}|$

choosing a morphism $f$ in a topological category $\mathbf{C}$ we want to construct a category a category $\mathrm{L}_f\mathbf{C}$ with the following property: given a morphism $F : \mathbf{C} \to \mathbf{D}$ in $\mathbf{Cat_{Top}}$ such that $F(f)$ is a weak equivalence in $\mathbf{D}$ then $F$ is factored (unique up to homotopy) as

$$\mathbf{C} \to \mathrm{L}_f\mathbf{C} \to \mathbf{D}.$$

**Lemma 8.5.** *The category $\mathrm{L}_f\mathbf{C}$ could be taken as following pushout in $\mathbf{Cat_{Top}}$:*

$$\begin{array}{ccc} \mathbf{A}^h & \xrightarrow{f} & \mathbf{C} \\ {\scriptstyle inc}\downarrow & & \downarrow{\scriptstyle i} \\ \mathbf{B}^h & \longrightarrow & \mathbf{M} \end{array}$$

*Moreover, $\pi_0\mathbf{C} \to \mathrm{L}_{\pi_0(f)}\pi_0\mathbf{C}$ is a localization in $\mathbf{Cat}$ and $\mathbf{C} \to \mathrm{L}_f\mathbf{C}$ is a cofibration in $\mathbf{Cat_{Top}}$.*

*Proof.* First, we note that the inclusion $inc$ is a cofibration in $\mathbf{Cat_{Top}}$. The functor $\mathbf{A}^h \to \mathbf{C}$ is constructed as follow: Let $\mathbf{A} \to \mathbf{C}$ be a functor which sends the only nontrivial morphism of $\mathbf{A}$ to $f \in \mathbf{C}$. It induces a map of simplicial sets $\mathrm{N}_\bullet\mathbf{A} \to \widetilde{\mathrm{N}_\bullet}\mathrm{sing}\mathbf{C}$ and by adjunction a functor $|\Xi \mathrm{N}_\bullet \mathbf{A}| \to \mathbf{C}$ which is the functor noted $f : \mathbf{A}^h \to \mathbf{C}$ in the diagram. The functor $inc : \mathbf{A}^h \to \mathbf{B}^h$ is induced by the functor $inc : \mathbf{A} \to \mathbf{B}$. Now suppose that we have a functor $\mathbf{C} \to \mathbf{D}$ which sends $f$ to a weak equivalence in $\mathbf{D}$. The induced functor $\mathbf{A}^h \to \mathbf{D}$ factors by $\mathbf{A}^h \to G\mathbf{D} \to \mathbf{D}$ where $G\mathbf{D}$ is the associated groupoid of $\mathbf{D}$ as seen in previews section.
Consider the diagram:

$$\begin{array}{ccc} \mathbf{A}^h & \longrightarrow & G\mathbf{D} \\ \downarrow{\scriptstyle inc} & & \downarrow{\scriptstyle i} \\ \mathbf{B}^h & \longrightarrow & \star \end{array}$$

and using the adjunctions we have a corresponding diagram in $\mathbf{sSet}$

$$\begin{array}{ccc} \mathrm{N}_\bullet\mathbf{A} & \longrightarrow & \widetilde{\mathrm{N}_\bullet}\mathrm{sing}G\mathbf{D} \\ \downarrow{\scriptstyle inc'} & & \downarrow{\scriptstyle i'} \\ \mathrm{N}_\bullet\mathbf{B} & \longrightarrow & \star \end{array}$$

But now $\mathrm{sing}G\mathbf{D}$ is a Kan complex see 6.16 and $inc'$ is a trivial cofibration in $\mathbf{sSet}$, so there exist a lifting (not unique) $\mathrm{N}_\bullet\mathbf{B} \to \mathrm{sing}G\mathbf{D}$. By adjunction we have a lifting $\mathbf{B}^h \to G\mathbf{D} \to \mathbf{D}$. So we can define unique morphism (up to homotopy) $\mathbf{M} \to \mathbf{D}$ and any functor $\mathbf{C} \to \mathbf{D}$ as before factors (uniquely up to homotopy) by $\mathbf{C} \to \mathbf{M} \to \mathbf{D}$. So a functorial model for $\mathrm{L}_f\mathbf{C}$ is $\mathbf{M}$ and the localization map $\mathbf{C} \to \mathrm{L}_f\mathbf{C}$ is a cofibration and in fact an inclusion of enriched categories. □

**Corollary 8.6.** *For any set $S$ of morphism in a topological category $\mathbf{C}$, the topological category $\mathrm{L}_S\mathbf{C}$ exist and it is unique up to homotopy. Moreover, the localization map $\mathbf{C} \to \mathrm{L}_S\mathbf{C}$ is a cofibration.*

## References

[1] I. Amrani. Categories simpliciales enrichies et k-theorie de waldhausen. *PhD thesis*, 2010.
[2] J.E. Bergner. A model category structure on the category of simplicial categories. *Transactions-American Mathematical Society*, 359(5):2043, 2007.
[3] W.G. Dwyer and K. Hess. Long knots and maps between operads. *Arxiv preprint arXiv:1006.0874*, 2010.
[4] W.G. Dwyer and D.M. Kan. Simplicial localizations of categories. *J. Pure Appl. Algebra*, 17(3):267–284, 1980.
[5] AD Elmendorf, I. Kriz, MA Mandell, JP May, and M. Cole. *Rings, modules, and algebras in stable homotopy theory.* American Mathematical Society, 2007.
[6] P.G. Goerss and JF Jardine. *Simplicial homotopy theory.* Birkhauser, 1999.
[7] A. Joyal. Advanced course on simplicial methods in higher categories. *CRM*, 2008.
[8] S. Mac Lane. Categories for the working mathematician. *Graduate Texts in Mathematics*, 1969.
[9] J. Lurie. Higher topos theory. *Arxiv preprint math.CT/0608040*, 2006.
[10] G. Segal. Categories and cohomology theories. *Topology*, 13(3):293–312, 1974.
[11] B. Toën. The homotopy theory of dg-categories and derived Morita theory. *Inventiones mathematicae*, 167(3):615–667, 2007.
[12] K. Worytkiewicz, K. Hess, P.E. Parent, and A. Tonks. A model structure à la Thomason on 2-Cat. *J. Pure Appl. Algebra*, 208(1):205–236, 2007.

Department of Mathematics, Masaryk University, Czech republic
*E-mail address*: ilias.amranifedotov@gmail.com
*E-mail address*: amrani@math.muni.cz