COMPARING COMMUTATIVE AND ASSOCIATIVE UNBOUNDED DIFFERENTIAL GRADED ALGEBRAS OVER $\mathbb{Q}$ FROM HOMOTOPICAL POINT OF VIEW

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Abstract. In this paper we establish a faithfulness result, in a homotopical sense, between a subcategory of the model category of augmented differential graded commutative algebras CDGA and a subcategory of the model category of augmented differential graded algebras DGA over the field of rational numbers $\mathbb{Q}$.

Introduction

It is well known that the forgetful functor from the category of commutative k-algebras to the category of category of associative k-algebras is fully faithful. We have an analogue result between the category of unbounded differential graded commutative k-algebras dgCAlg_k and the category of unbounded differential graded associative algebras dgAlg_k. The question that we want explore is the following: Suppose that $k = \mathbb{Q}$, is it true that forgetful functor $U : \mathsf{dgCAlg}_k \to \mathsf{dgAlg}_k$ induces a fully faithful functor at the level of homotopy categories

$$\mathbf{R}U : \mathrm{Ho}(\mathsf{dgCAlg}_k) \to \mathrm{Ho}(\mathsf{dgAlg}_k).$$

The answer is **no**. A nice and easy counterexample was given by Lurie. He has considered $k[x, y]$ the free commutative CDGA in two variables concentrated in degree 0. It follows obviously that

$$\mathrm{Ho}(\mathsf{dgCAlg}_k)(k[x, y], S) \simeq \mathrm{H}^0(S) \oplus \mathrm{H}^0(S),$$

while

$$\mathrm{Ho}(\mathsf{dgAlg}_k)(k[x, y], S) \simeq \mathrm{H}^0(S) \oplus \mathrm{H}^0(S) \oplus \mathrm{H}^{-1}(S).$$

Something nice happens if we consider the category of augmented CDGA denoted by dgCAlg_k^* and augmented DGA denoted by dgAlg_k^*.

Theorem 0.1 (3.1). *For any R and S in dgCAlg_k^*, the induced map by the forgetful functor*

$$\Omega\mathrm{Map}_{\mathsf{dgCAlg}_k^*}(R, S) \to \Omega\mathrm{Map}_{\mathsf{dgAlg}_k^*}(R, S),$$

has a retract, in particular

$$\pi_i\mathrm{Map}_{\mathsf{dgCAlg}_k^*}(R, S) \to \pi_i\mathrm{Map}_{\mathsf{dgAlg}_k^*}(R, S)$$

is injective $\forall\ i > 0$.

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Let S be a differential graded commutative algebra which is a "loop" of an other CDGA algebra A, i.e. $S = \mathsf{Holim}(k \to A \leftarrow k)$, where the homotopy limit is taken in the model category dgCAlg_k. A direct consequence of our theorem is that the right derived functor $\mathbf{R}U$ is a faithful functor i.e., the induced map $\mathrm{Ho}(\mathsf{dgCAlg}_k^*)(R,S) \to \mathrm{Ho}(\mathsf{dgAlg}_k^*)(R,S)$ is injective.

Interpretation of the result in the derived algebraic geometry. Rationally, any pointed topological X space can be viewed as an augmented (connective) commutative differential graded algebra via its cochain complex $C^*(X,\mathbb{Q})$. In case where X is a simply connected rational space, the cochain complex $C^*(X,\mathbb{Q})$ carries the whole homotopical information about X, by Sullivan Theorem [5]. Moreover, the bar construction $\mathrm{B}C^*(X,\mathbb{Q})$ is identified (as E_∞-DGA) to $C^*(\Omega X,\mathbb{Q})$ and $\Omega C^*(X,\mathbb{Q})$ is identified (as E_∞-DGA) to $C^*(\Sigma X,\mathbb{Q})$ cf. [4]. This interpretation allows us to make the following definition: A generalized rational pointed space is an augmented commutative differential graded $\mathbb{Q}$-algebra (possibly unbounded). In the same spirit, we define a pointed generalized **noncommutative rational space** as an augmented differential graded $\mathbb{Q}$-algebra (possibly unbounded). Let A be any augmented CDGA resp. DGA, we will call a CDGA resp. DGA of the form ΩA a *op-suspended* CDGA resp. DGA. Our theorem 3.1, can be interpreted as follows: **The homotopy category of op-suspended generalized commutative rational spaces is a subcategory of the homotopy category of op-suspended generalized noncommutative rational spaces.**

1. DGA, CDGA and E_∞-DGA.

We work in the setting of unbounded differential graded k-modules dgMod_k. This is a a symmetric monoidal closed model category (k is a commutative ring). We denote the category of (reduced) operads in dgMod_k by Op_k. We follow notations and definitions of [2], we say that an operad P is *admissible* if the category of $\mathsf{P}-\mathsf{dgAlg}_k$ admits a model structure where the fibrations are degree wise surjections and weak equivalence are quasi-isomorphisms. For any map of operads $\phi : \mathsf{P} \to \mathsf{Q}$ we have an induced adjunction of the corresponding categories of algebras:

$$\mathsf{P} - \mathsf{dgAlg}_k \underset{\phi^*}{\overset{\phi_!}{\rightleftarrows}} \mathsf{Q} - \mathsf{dgAlg}_k.$$

A Σ-cofibrant operad P is an operad such that $\mathsf{P}(n)$ is $k[\Sigma_n]$-cofibrant in $\mathsf{dgMod}_{k[\Sigma_n]}$. Any cofibrant operad P is a Σ-cofibrant operad [2, Proposition 4.3]. We denote the associative operad by Ass and the commutative operad by Com.The operad Ass is an admissible operad and Σ-cofibrant, while the operad Com is not admissible in general. In the rational case, when $k = \mathbb{Q}$ the operad Com is admissible but not Σ-cofibrant. More generally any cofibrant operad P is admissible [2, Proposition 4.1, Remark 4.2]. We define a symmetric tensor product of operads by the formulae

$$[\mathsf{P} \otimes \mathsf{Q}](n) = \mathsf{P}(n) \otimes \mathsf{Q}(n), \ \forall \ n \in \mathbb{N}.$$

Lemma 1.1. *Suppose that* $\phi : \mathsf{Ass} \to \mathsf{P}$ *is a cofibration of operads.The operad* P *is admissible and the functor* $\phi^* : \mathsf{P} - \mathsf{dgAlg}_k \to \mathsf{dgAlg}_k$ *preserves fibrations, weak equivalences and cofibrations with cofibrant domain in the inderleing category* dgMod_k*.*

Proof. First of all, the operad P is admissible, indeed we use the cofibrant resolution $r : \mathsf{E}_\infty \to \mathsf{Com}$ and consider the following pushout in Op_k given by:

$$\begin{array}{ccc} \mathsf{Ass}_\infty & \hookrightarrow & \mathsf{E}_\infty \\ \downarrow{\sim} & & \downarrow{\alpha} \\ \mathsf{Ass} & \xrightarrow{f} & \mathsf{E}'_\infty \end{array}$$

Where Ass_∞ in the cofibrant replacement of Ass in Op_k and $\mathsf{Ass}_\infty \to \mathsf{E}_\infty$ is a cofibration. Since the category Op_k is left proper in the sense of [8, Theorem 3], we have that $\alpha : \mathsf{E}_\infty \to \mathsf{E}'_\infty$ is an equivalence. We denote by I the unit interval in the category dgMod_k which is strictly coassociative. The opposite endomorphism operad $\mathsf{End}^{op}(I)$ has a structure of E_∞-algbra and Ass_∞-algebra which factors through Ass i.e., we have two compatible maps of operads:

$$\begin{array}{ccccc} \mathsf{Ass}_\infty & \hookrightarrow & \mathsf{E}_\infty & & \\ \downarrow{\sim} & & \downarrow{\alpha} & \searrow & \\ \mathsf{Ass} & \overset{f}{\hookrightarrow} & \mathsf{E}'_\infty & \dashrightarrow & \\ & \searrow & & & \mathsf{End}^{op}(I) \end{array}$$

by the universality of the pushout, we have a map of operads $\mathsf{E}'_\infty \to \mathsf{End}^{op}(I)$. This means that the unit interval I has a structure of E'_∞-colagebra [2, p.4]. Moreover, we have a commutative diagram in Op_k given by

$$\begin{array}{ccccc} \mathsf{Ass} & \xrightarrow{\Delta} & \mathsf{Ass} \otimes \mathsf{Ass} & \xrightarrow{\phi\otimes f} & \mathsf{P} \otimes \mathsf{E}'_\infty \\ \downarrow{\phi} & & & & \downarrow{id\otimes r}\ {\sim} \\ \mathsf{P} & & \xrightarrow{\quad id \quad} & & \mathsf{P} \otimes \mathsf{Com} = \mathsf{P} \end{array}$$

where the diagonal map $\Delta : \mathsf{Ass} \to \mathsf{Ass} \otimes \mathsf{Ass}$ is induced by the diagonals $\Sigma_n \to \Sigma_n \times \Sigma_n$. Hence, the map $\mathsf{P} \otimes \mathsf{E}'_\infty \to \mathsf{P}$ admits a section. It implies by [2, Proposition 4.1], that P is admissible and Σ-cofibrant. Since all objects in $\mathsf{P} - \mathsf{dgAlg}_k$ are fibrant and ϕ^* is a right Quillen adjoint, it preserves fibrations and weak equivalences.

Since P is an admissible operad, we have a Quillen adjunction

$$\mathsf{dgAlg}_k \underset{\phi^*}{\overset{\phi_!}{\rightleftarrows}} \mathsf{P} - \mathsf{dgAlg}_k,$$

where the functor ϕ^* is identified to the forgetful functor. Moreover, the model structure on $\mathsf{P} - \mathsf{dgAlg}_k$ is the transfered model structure from the cofibrantly generated model structure dgAlg_k via the adjunction $\phi_!, \phi^*$. Suppose that $f : A \to C$ is a cofibration in $\mathsf{P} - \mathsf{dgAlg}_k$ such that A is cofibrant in dgMod_k. We factor this map as a cofibration followed by a trivial fibration

$$A \overset{i}{\hookrightarrow} P \underset{\sim}{\overset{p}{\twoheadrightarrow}} B$$

in the category dgAlg_k. By [7, Lemma 4.1.16], we have an induced map of endomorphism operads (of diagrmas):

$$\mathrm{End}_{\{A\to P\to B\}} \to \mathrm{End}_{\{A\to B\}}$$

which is a trivial fibration. Moreover, we have the following commutative diagram in Op_k

$$\begin{array}{ccc} \mathsf{Ass} & \longrightarrow & \mathrm{End}_{\{A\to P\to B\}} \\ \big\downarrow & \nearrow & \big\downarrow \sim \\ \mathsf{P} & \longrightarrow & \mathrm{End}_{\{A\to B\}} \end{array}$$

Since Op_k is a model category, it implies that we have a lifting map of operads $\mathsf{P} \to \mathrm{End}_{\{A\to P\to B\}}$, hence i and p are maps of $\mathsf{P}-\mathsf{dgAlg}_k$. Therefore, we consider the following commutative square in the category $\mathsf{P}-\mathsf{dgAlg}_k$

$$\begin{array}{ccc} A & \xrightarrow{i} & P \\ f\big\downarrow & \overset{r}{\nearrow} & \big\downarrow p\ \sim \\ B & \xrightarrow{id} & B \end{array}$$

the lifting map r exists since $\mathsf{P}-\mathsf{dgAlg}_k$ is a model category, we conclude that $p\circ r = id$ and $r\circ f = i$, which means that f is a retract of i, hence f is a cofibration in dgAlg_k. □

Remark 1.2. With the same notation as in 1.1, if A is a cofibrant object in $\mathsf{P}-\mathsf{dgAlg}_k$ then A is a cofibrant object in dgMod_k. Indeed $k\to A$ is a cofibration in $\mathsf{P}-\mathsf{dgAlg}_k$, by the previous lemma $k\to A$ is a cofibration in dgAlg_k. Therefore, $k\to A$ is a cofibration in dgMod_k.

2. Suspension in CGDA and DGA

We denote the the operad $\mathsf{E}^{'}_\infty$ of the previous section by E_∞, and $k=\mathbb{Q}$.

2.1. E_∞-DGA. We have a map of operads $\mathsf{Ass}\to\mathsf{Com}$, which we factor as cofibration followed by a trivial fibration.

$$\mathsf{Ass} \hookrightarrow \mathsf{E}_\infty \overset{\sim}{\twoheadrightarrow} \mathsf{Com}$$

As a consequence, we have the following Quillen adjunctions

$$\mathsf{dgAlg}_k \underset{U}{\overset{Ab_\infty}{\rightleftarrows}} \mathsf{E}_\infty\mathsf{dgAlg}_k \underset{U^{'}}{\overset{str}{\rightleftarrows}} \mathsf{dgCAlg}_k$$

These adjunctions have the following properties:

- The functors $U^{'}$ and $U\circ U^{'}$ and are the forgetful functors, they are fully faithful cf 2.3 and 2.2.
- The functors str, $U^{'}$ form a Quillen equivalence since $k=\mathbb{Q}$ cf [6, Corollary 1.5]. The functor str is the strictification functor.
- The functors Ab_∞, U form a Quillen pair.
- The composition $str\circ Ab_\infty$ is the abelianization functor $Ab : \mathsf{dgAlg}_k \to \mathsf{dgCAlg}_k$.
- The functors str and Ab are idenpotent functors. cf 2.3 and 2.2.

The model categories dgCAlg_k^* and dgAlg_k^* and $\mathsf{E}_\infty\mathsf{dgAlg}_k^*$ are pointed model categories. It is natural to introduce the suspension functors in these categories.

Definition 2.1. Let C be any pointed model category, we denote the point by 1, and let $A \in \mathsf{C}$, a suspension ΣA is defined as $\mathsf{hocolim}(1 \leftarrow A \rightarrow 1)$.

Proposition 2.2. Any map $f : A \rightarrow S$ in $\mathsf{E}_\infty\mathsf{dgAlg}_k$, where S is in dgCAlg_k factors in a unique way as $A \rightarrow str(A) \rightarrow S$ and the forgetful functor $U^{'} : \mathsf{dgCAlg}_k \rightarrow \mathsf{E}_\infty\mathsf{dgAlg}_k$ is fully faithful. Moreover, the unit of the adjunction $\nu_A : A \rightarrow str(A)$ is a fibration.

Proof. Suppose that we have a map $h : R \rightarrow S$ in $\mathsf{E}_\infty\mathsf{dgAlg}_k$ such that R and S are objects in dgCAlg_k. By definition of the operad E_∞ the map h has to be associative, therefore h is a morphism in dgCAlg_k since R and S are commutative differential graded algebras. The forgetful functor $U^{'} : \mathsf{dgCAlg}_k \rightarrow \mathsf{E}_\infty\mathsf{dgAlg}_k$ is fully faithful, this implies that $str(S) = S$ for any $S \in \mathsf{dgCAlg}_k$. We have a commutative diagram induced by the unit ν of the adjunction $(U^{'},\ str)$:

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & S \\ {\scriptstyle \nu_A}\downarrow & & \downarrow{\scriptstyle \nu_S = id} \\ str(A) & \xrightarrow{str(f)} & str(S) = S. \end{array}$$

We conclude that $f = str(f) \circ \nu_A$. The surjectivity of the ν_A follows from the universal property of $str(A)$. Hence, ν_A is a fibration in $\mathsf{E}_\infty\mathsf{dgAlg}_k$. □

Proposition 2.3. Any map $f : A \rightarrow S$ in dgAlg_k, where S is in dgCAlg_k factors in a unique way as $A \rightarrow Ab(A) \rightarrow S$ and the forgetful functor $U \circ U^{'} : \mathsf{dgCAlg}_k \rightarrow \mathsf{dgAlg}_k$ is fully faithful. Moreover, the unit of the adjunction $\nu_A : A \rightarrow Ab(A)$ is a fibration.

Proof. The proof is the same as in 2.2. □

Proposition 2.4. Suppose that we have a trivial cofibration $k \rightarrow \underline{k}$ in $\mathsf{E}_\infty\mathsf{dgAlg}_k$. Then the universal map $\pi :\ Ab(\underline{k}) \rightarrow str(\underline{k})$ is a trivial fibration and admits a section in the category dgCAlg_k.

Proof. We consider the following commutative diagram in $\mathsf{E}_\infty\mathsf{dgAlg}_k$:

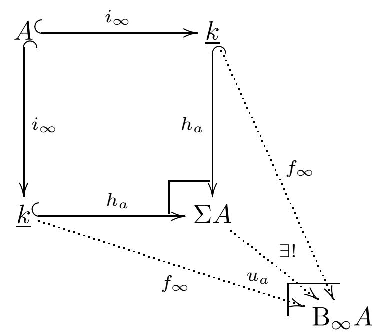
$$\begin{array}{ccc} k & \xrightarrow{\ \sim\ } & \underline{k} \\ {\scriptstyle id}\downarrow & & \downarrow \\ k = str(k) & \xrightarrow{\ \sim\ } & str(\underline{k}). \end{array}$$

The map $k \rightarrow str(\underline{k})$ is an equivalence since str is left Quillen functor, the same thing holds for the abelianization functor i.e., $\underline{k} \rightarrow Ab(\underline{k})$ is a trivial fibration, since $k \rightarrow \underline{k}$ is a trivial cofibration in dgAlg_k 1.1 and Ab is a left Quillen functor. On another hand the map $\underline{k} \rightarrow str(\underline{k})$, which is a trivial fibration in $\mathsf{E}_\infty\mathsf{dgAlg}_k$ and hence in dgAlg_k, can be factored (cf 2.3) as $\underline{k} \rightarrow Ab(\underline{k}) \rightarrow str(\underline{k})$, where $Ab(\underline{k}) \rightarrow str(\underline{k})$ is a trivial fibration between cofibrant object in dgCAlg_k. It follows that we have a retract $l : str(\underline{k}) \rightarrow Ab(\underline{k})$. □

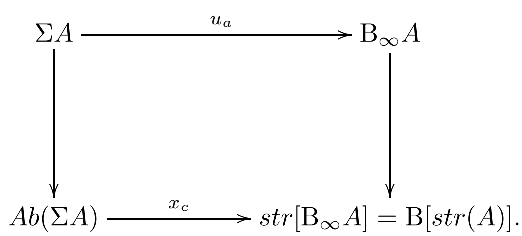
Definition 2.5. The suspension functor in the pointed model categories dgCAlg_k^*, dgAlg_k^* and $\mathsf{E}_\infty\mathsf{dgAlg}_k^*$ are denoted by B, Σ and B_∞.

Lemma 2.6. *Suppose that A is a cofibrant object in $\mathsf{E}_\infty\mathsf{dgAlg}_k^*$, and $i : A \to \underline{k}$ a cofibration, then $str(\mathrm{B}_\infty A)$ is a retract of $Ab(\Sigma A)$ in the category dgCAlg_k.*

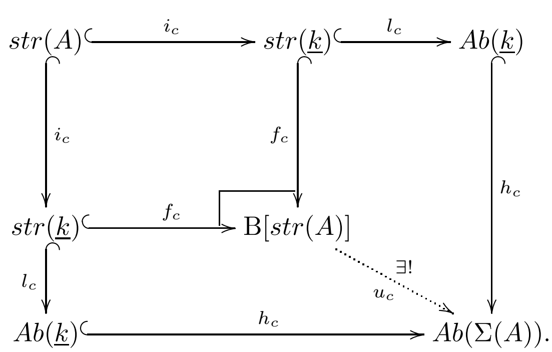
Proof. First of all if a map f is associative, commutative resp. E_∞-map we put an index f_a, f_c resp. f_∞, notice that by definition of the operad E_∞ any E_∞-map is a strictly associative map. Suppose that A is a cofibrant object in $\mathsf{E}_\infty\mathsf{dgAlg}_k$. Consider the following commutative square:



where ΣA is the (homotopy 1.1) pushout in dgAlg_k and $\mathrm{B}_\infty A$ is the (homotopy) pushout in $\mathsf{E}_\infty\mathsf{dgAlg}_k$. By proposition 2.2 and proposition 2.3 we have a following commutative square in dgAlg_k:



By 2.4 we have an inclusion of commutative differential graded algebras $l_c : str(\underline{k}) \to Ab(\underline{k})$ and after strictification we obtain on another (homotopy) pushout square in dgCAlg_k given by



In order to prove that $\mathrm{B}[str(A)]$ is a retract of $Ab(\Sigma(A))$ it is sufficient to prove that

$$x_c \circ h_c \circ l_c = f_c.$$

By proposition 2.2 and proposition 2.3, the composition E_∞-maps

$$\underline{k} \xrightarrow{f_\infty} \mathrm{B}_\infty A \longrightarrow str[\mathrm{B}_\infty A]$$

can be factored in a unique way as

$$\underline{k} \longrightarrow Ab(\underline{k}) \xrightarrow{\pi} str(\underline{k}) \xrightarrow{\alpha_c} str[\mathrm{B}_\infty A] = \mathrm{B}[str(A)].$$

By unicity, $\alpha_c = f_c$. On another hand, using the first poushout in $\mathsf{E}_\infty\mathsf{dgAlg}_k$, the previous composition $\underline{k} \to str[\mathrm{B}_\infty A]$ is factored as

$$\underline{k} \xrightarrow{h_a} \Sigma A \longrightarrow Ab(\Sigma A) \xrightarrow{x_c} str[\mathrm{B}_\infty A].$$

We summarize the previous remarks in the following commutative diagram:

$$\begin{array}{ccccccc}
\underline{k} & \xrightarrow{pr} & Ab(\underline{k}) & \xrightarrow{\pi} & str(\underline{k}) & \xrightarrow{f_c} & str[\mathrm{B}_\infty A] \\
\downarrow{\scriptstyle id} & & \vdots{\scriptstyle h_c} & & & & \downarrow{\scriptstyle id} \\
\underline{k} & \longrightarrow & Ab(\Sigma A) & & \xrightarrow{\quad x_c \quad} & & str[\mathrm{B}_\infty A]
\end{array}$$

by definition of h_a, the doted map h_c makes the left square commutative. Since the whole square is commutative and the map pr is surjective we conclude that $x_c \circ h_c = f_c \circ \pi$. Since the map $l_c : Str(\underline{k}) \to Ab(\underline{k})$ is a retract of π (Cf. 2.4) i.e., $\pi \circ l_c = id$, we conclude that $x_c \circ h_c \circ l_c = f_c$. Finally, by unicity of the pushout, we deduce that the following composition

$$\mathrm{B}[str(A)] \xrightarrow{u_c} Ab(\Sigma A) \xrightarrow{x_c} \mathrm{B}[str(A)]$$

is identity. □

3. Main result and applications

Theorem 3.1. *For any R and S in dgCAlg_k^*, the induced map by the forgetful functor*

$$\Omega\mathrm{Map}_{\mathsf{dgCAlg}_k^*}(R,S) \to \Omega\mathrm{Map}_{\mathsf{dgAlg}_k^*}(R,S),$$

has a retract, in particular

$$\pi_i\mathrm{Map}_{\mathsf{dgCAlg}_k^*}(R,S) \to \pi_i\mathrm{Map}_{\mathsf{dgAlg}_k^*}(R,S)$$

is injective $\forall\ i > 0$.

Proof. Suppose that R is (cofibrant) object in $\mathsf{E}_\infty\mathsf{dgAlg}_k$ and S any object in dgCAlg_k. By adjunction, we have that

$$\begin{aligned}
\Omega\mathrm{Map}_{\mathsf{dgCAlg}_k^*}(str(R),S) &\sim \mathrm{Map}_{\mathsf{dgCAlg}_k^*}(\mathrm{B}[str(R)],S) &(3.1)\\
&\sim \mathrm{Map}_{\mathsf{dgCAlg}_k^*}(str[\mathrm{B}_\infty R],S) &(3.2)\\
&\sim \mathrm{Map}_{\mathsf{E}_\infty\mathsf{dgAlg}_k^*}(\mathrm{B}_\infty R,S) &(3.3)\\
&\sim \Omega\mathrm{Map}_{\mathsf{E}_\infty\mathsf{dgAlg}_k^*}(R,S). &(3.4)
\end{aligned}$$

By Lemma 2.6, we have a retract

$$\mathrm{Map}_{\mathsf{dgCAlg}_k^*}(\mathrm{B}[str(R)],S) \to \mathrm{Map}_{\mathsf{dgCAlg}_k^*}(Ab(\Sigma R),S) \to \mathrm{Map}_{\mathsf{dgCAlg}_k^*}(\mathrm{B}[str(R)],S).$$

Again by adjunction:

$$\mathrm{Map}_{\mathsf{dgCAlg}_k^*}(Ab(\Sigma R),S) \sim \mathrm{Map}_{\mathsf{dgAlg}_k^*}(\Sigma R,S) \sim \Omega\mathrm{Map}_{\mathsf{dgAlg}_k^*}(R,S).$$

We conclude that

$$\Omega \mathrm{Map}_{\mathsf{E}_\infty \mathsf{dgAlg}_k^*}(R,S) \xrightarrow{U} \Omega \mathrm{Map}_{\mathsf{dgAlg}_k^*}(R,S) \longrightarrow \Omega \mathrm{Map}_{\mathsf{E}_\infty \mathsf{dgAlg}_k^*}(R,S)$$

is a retract. Hence, the forgetful functor U induces a injective map on homotopy groups i.e.,

$$\pi_i \mathrm{Map}_{\mathsf{dgCAlg}_k^*}(str(R),S) \simeq \pi_i \mathrm{Map}_{\mathsf{E}_\infty \mathsf{dgAlg}_k^*}(R,S) \to \pi_i \mathrm{Map}_{\mathsf{dgAlg}_k^*}(R,S)$$

is injective $\forall\ i > 0$. □

3.1. **Rational homotopy theory.** We give an application of our theorem 3.1 in the context of rational homotopy theory. Let X be a simply connected rational space such that $\pi_i X$ is finite dimensional $\mathbb{Q}$-vector space for each $i > 0$. Let $C^*(X)$ be the differential graded $\mathbb{Q}$-algebra cochain associated to X which is a connective $\mathsf{E}_\infty\mathsf{dgAlg}_k$. By Sullivan theorem $\pi_i X \simeq \pi_i \mathrm{Map}_{\mathsf{dgCAlg}_k^*}(C^*(X),\mathbb{Q})$. By 3.1, we have that $\pi_i X$ is a sub $\mathbb{Q}$-vector space of $\pi_i \mathrm{Map}_{\mathsf{dgAlg}_k^*}(R,S)$. On another hand [1], since $C^*(X)$ is connective, we have that for any $i > 1$

$$\pi_i \mathrm{Map}_{\mathsf{dgAlg}_k^*}(C^*(X),\mathbb{Q}) \simeq \mathrm{HH}^{-1+i}(C^*(X),\mathbb{Q}),$$

where HH^* is the Hochschild cohomology. Since we have assumed finiteness condition on X, we have that

$$\mathrm{HH}^{-1+i}(C^*(X),\mathbb{Q}) \simeq \mathrm{HH}_{i-1}(C^*(X),\mathbb{Q}).$$

The functor $C^*(-,\mathbb{Q}) : \mathsf{Top}^{op} \to \mathsf{E}_\infty\mathsf{dgAlg}_k$ commutes with finite homotopy limits, where Top is the category of simply connected spaces. Hence,

$$\mathrm{HH}_{-1+i}(C^*(X),\mathbb{Q}) = \mathrm{H}^{i-1}[C^*(X) \otimes^{\mathbf{L}}_{C^*(X\times X)} \mathbb{Q}] \simeq \mathrm{H}^{i-1}(\Omega X,\mathbb{Q}).$$

We conclude that $\pi_i X$ is a sub $\mathbb{Q}$-vector space of $\mathrm{H}^{i-1}(\Omega X,\mathbb{Q})$.
More generally by Block-Lazarev result [3] on rational homotopy theory and [1], we have an injective map of $\mathbb{Q}$-vector spaces

$$\mathrm{AQ}^{-i}(C^*(X),C^*(Y)) \to \mathrm{HH}^{-i+1}(C^*(X),C^*(Y)),$$

where the $C^*(X)$-(bi)modules structure on $C^*(Y)$ is given by $C^*(X) \to \mathbb{Q} \to C^*(Y)$, and AQ^* is the André-Quillen cohomology. We also assume that X and Y are simply connected and $i > 1$.
More generally,

$$\pi_i \mathrm{Map}_{\mathsf{E}_\infty \mathsf{dgAlg}_k}(R,S) = \mathrm{AQ}^{-i}(R,S) \to \mathrm{HH}^{-i+1}(R,S) = \pi_i \mathrm{Map}_{\mathsf{dgAlg}_k}(R,S)$$

is an injective map of $\mathbb{Q}$-vector spaces for all $i > 1$ and any augmented E_∞-differential graded connective $\mathbb{Q}$-algebras R and S, where the action of S on R is given by $S \to \mathbb{Q} \to R$.

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