THE MAPPING SPACE OF UNBOUNDED DIFFERENTIAL GRADED ALGEBRAS

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Abstract. In this paper, we give a concrete description of the higher homotopy groups ($n > 0$) of the mapping space $\mathrm{Map}_{\mathsf{dgAlg}_k}(R,S)$ for R and S unbounded differential graded algebras (DGA) over a commutative ring k. In the connective case, we describe the relation between the higher (negative) Hochschild cohomology $\mathrm{HH}_k^{-n+1}(R,S)$ and higher homotopy groups $\pi_n\mathrm{Map}_{\mathsf{dgAlg}_k}(R,S)$, when $n > 1$.

Introduction

Our work is based on the recent paper [4]. Given a (symmetric) monoidal model category $(\mathsf{C},\otimes)$ and a compatible model structure on the category of monoids $\mathsf{C}^{\otimes}$ (with underlying weak equivalences and fibrations of C). Suppose that we have a Dwyer-Kan model structure on the category of small C-enriched categories $\mathsf{Cat}_{\mathsf{C}}$ (weak equivalences are homotopy enriched fully faithful and homotopy essentially surjective functors). The main idea is that the mapping spaces Map of these three model structures are closely related under some assumptions (**six Axioms** (2) described in [4, section 3]). The none obvious axiom is the third one 2.4. Our paper is a concrete application of this idea in the setting, where

- C is the symmetric monoidal category of **unbounded differential graded modules** (DG modules for short) over a commutative ring k [7].
- $\mathsf{C}^{\otimes}$ is the model category of unbounded DG k-algebras [11].
- $\mathsf{Cat}_{\mathsf{C}}$ is the model category of (small) DG categories, denoted by dg-Cat [13].

Before going to our main point, we illustrate the previous idea in the *topological* setting when $\mathsf{C} = \mathsf{Top}$. There is a well known fiber sequence (Cf. [4, p. 6]) for given topological groups G and H, which is

$$\mathrm{map}_*(BH,BG) \to \mathrm{map}(BH,BG) \to BG, \tag{0.1}$$

where BG and BH are the classifying spaces of G and H. Notice that if G and H are discrete groups, then $\pi_0\mathrm{map}(BH,BG) = \mathrm{Rep}_G(H)$ is the set of equivalence classes of representations of H in G. The interpretation in the categorical setting is as follows. Let $\mathsf{Cat}_{\mathsf{Top}}$ be the model category of small topological categories [2], where weak equivalences are Dwyer-Kan equivalences. Denote by $\mathsf{Top}^{\otimes}$ the model

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category of topological monoids. The previous fiber sequence (0.1) has a model-categorical translation in terms of mapping spaces:

$$\mathrm{Map}_{\mathsf{Top}^{\otimes}}(H,G) \to \mathrm{Map}_{\mathsf{Cat}_{\mathsf{Top}}}(\mathsf{H},\mathsf{G}) \to \mathrm{Map}_{\mathsf{Cat}_{\mathsf{Top}}}(*,\mathsf{G}), \tag{0.2}$$

where G (resp. H) is the topological category with one object and endomorphism monoid G (resp H). Here, we had supposed that G and H are topological groups. The fiber sequence (0.2) is still valid for topological monoids. and coincides with 0.1 in the case where G and H are topological groups.

The goal in this paper is the construction of fiber sequence (0.2) in the unbounded differential graded setting (Cf. 3.3), namely

$$\mathrm{Map}_{\mathsf{dgAlg}_k}(R,S) \to \mathrm{Map}_{dg-\mathsf{Cat}}(\mathsf{R},\mathsf{S}) \to \mathrm{Map}_{dg-\mathsf{Cat}}(\mathsf{k},\mathsf{S}). \tag{0.3}$$

Notations

In what follows, all model structures are defined by taking homology isomorphisms for weak equivalences and degree-wise surjections for fibrations.

- k is a fixed commutative ring of any characteristic.
- dgMod_k the stable symmetric monoidal closed model category of differential graded k-modules. Our convention is the cohomological gradation i.e., the differentials increase the degree by $+1$.
- The (**derived when necessary**) tensor product over k of differential graded k-modules is denoted by $\otimes$.
- dgAlg_k is the model category of unbounded differential graded k-algebras [11] i.e., the category of monoids in dgMod_k.
- dgAlg_R (differential graded R-algebras) is the model category of graded differential k-algebras under a fixed graded differential k-algebra R, i.e., objects are morphisms $R \to A$ in dgAlg_k.
- dgAlg_{R-S} is the model category of graded differential k-algebras under a fixed graded differential k algebras R and S i.e. objects are pairs of morphisms $R \to A,\ S \to A$ in dgAlg_k.
- For any differential graded k-algebras R and S we denote by dgMod_{R-S} the stable model category of differential graded $R-S$-bimodules. The category dgMod^0_{R-S} is the model category of pointed DG $R-S$-modules i.e., objects are coming with an extra map $k \to M$.
- We denote the **derived** mapping space of a model category by Map.
- The n-th homology group of a differential graded k-algebra R is denoted by $\mathrm{H}^n R$
- The suspension functor $\Sigma : \mathsf{dgMod}_R \to \mathsf{dgMod}_R$ is defined as follows $(\Sigma M)_n = M_{n+1}$. Obviously, this functor has an inverse denoted by Σ^{-1}.
- Let $R \in \mathsf{dgAlg}_k$, we denote the derived category of R by D_R which is the homotopy category of DG R-modules, i.e., $\mathrm{Ho}(\mathsf{dgMod}_R)$. For more details Cf. [7].

0.1. **Ext functor and Hochschild cohomology.** In this paragraph, we recall a well known translation between notions defined in *Algebraic Geometry* and *Algebraic Topology*. We use the same conventions as in [6]. Let $R \in \mathsf{dgAlg}_k$, and

$M, N \in \mathsf{dgMod}_R$. In the stable model category of dgMod_R [7, 8] and for $n \in \mathbb{Z}$,

$$\begin{aligned}\mathrm{Ext}^n_R(N, M) &\simeq \mathsf{D}_R(N, \Sigma^n M)\\ &\simeq \mathrm{Ho}(\mathsf{dgMod}_R)(N, \Sigma^n M)\\ &\simeq \pi_0 \mathrm{Map}_{\mathsf{dgMod}_R}(N, \Sigma^n M).\end{aligned}$$

Remark 0.1. Our gradation is the same as in [8] and opposite to the one used in [7].

Remark 0.2. Recall that the model category dgMod_R is naturally pointed, and the functor Σ^{-1} is the loop functor. Therefore, it follows by [7, Lemma 6.1.2], that for any $n \geq 0$ there is a weak homotopy equivalence of pointed simplicial sets, where the base point is the zero morphism,

$$\mathrm{Map}_{\mathsf{dgMod}_R}(M, \Sigma^{-n} N) \sim \Omega^n \mathrm{Map}_{\mathsf{dgMod}_R}(M, N).$$

For $n \geq 0$, we have the following group isomorphisms:

$$\begin{aligned}\mathrm{Ext}^{-n}_R(N, M) &\simeq \pi_0 \mathrm{Map}_{\mathsf{dgMod}_R}(N, \Sigma^{-n} M)\\ &\simeq \pi_n \mathrm{Map}_{\mathsf{dgMod}_R}(N, M).\end{aligned}$$

Remark 0.3. If R is a k-algebra and $M,\ N$ are any two R-modules i.e., R, M and N are DG modules concentrated in degree 0, then $\mathrm{Ext}^n_R(M, N) = 0$ for $n < 0$ (Cf. [8]).

Definition 0.4. Let $R \in \mathsf{dgAlg}_k$ and let M be a DG R-bimodule. The Hochschild cohomology of R with coefficient in M is defined for all $n \in \mathbb{Z}$, by

$$\begin{aligned}\mathrm{HH}^n_k(R, M) &\simeq \mathrm{Ext}^n_{R\otimes R^{op}}(R, M)\\ &\simeq \pi_0 \mathrm{Map}_{\mathsf{dgMod}_{R-R}}(R, \Sigma^n M).\end{aligned}$$

If $M = R$, we denote the Hochschild cohomology of R with coefficient in R simply by $\mathrm{HH}^*_k(R)$.

Remark 0.5. The correct definition for the Hochschild cohomology $\mathrm{HH}^n_k(R, M)$ is $\mathrm{Ext}^n_{R\otimes^{\mathsf{L}} R^{op}}(R, M)$, but we took the liberty to denote the derived tensor product as an ordinary tensor product!

Remark 0.6. For any DG R-module N, we recall that (Cf. [8])

$$\pi_n \mathrm{Map}_{\mathsf{dgMod}_R}(R, N)_* \simeq \mathsf{D}_R(R, \Sigma^{-n} N) \simeq \mathrm{H}^{-n}(N).$$

Remark 0.7. When the homotopy groups are computed without mentioning the base point, it will mean that the base point is the null morphism.

For any DG algebra R and any R-bimodule M, the usual definition of the Hochschild cohomology is given by $\mathrm{HH}^*_k(R, M) = \mathrm{H}^* \mathrm{Hom}_{R\otimes R^{op}}(R, M)$ (e.g. [3, 3.14]), where $\mathrm{Hom}_{R\otimes R^{op}}(R, -) : \mathsf{dgMod}_{R-R} \to \mathsf{dgMod}_k$, is the right (derived) functor having as left (derived) adjoint $R \otimes -$. If $n \geq 0$, applying [4, Theorem 2.12], we obtain the following group isomorphisms

$$\begin{aligned}\mathrm{H}^{-n} \mathrm{Hom}_{R\otimes R^{op}}(R, M) &\simeq \pi_n \mathrm{Map}_{\mathsf{dgMod}_k}(k, \mathrm{Hom}_{R\otimes R^{op}}(R, M))\\ &\simeq \pi_n \mathrm{Map}_{\mathsf{dgMod}_{R-R}}(R, M)\\ &\simeq \mathrm{Ext}^{-n}_{R\otimes R^{op}}(R, M).\end{aligned}$$

Remark 0.8. In order to be clear, by derived functor of $\mathrm{Hom}_{R\otimes R^{op}}(R, M)$ we mean the right derived functor $\mathbf{R}\mathrm{Hom}_{R\otimes R^{op}}(R, M)$.

1. Main results

We start by fixing a morphism $\phi : R \to S$ in dgAlg_k (such that S is a strict DG k-algebra 2.9). Our main result concerns the higher homotopy groups of the mapping space

$$\pi_n \mathrm{Map}_{\mathsf{dgAlg}_k}(R, S)_\phi := [R, S]^{\otimes}_n \text{ for } n > 1.$$

We give an explicit long exact sequence relating these higher homotopy groups with $\mathrm{H}^* S$ and (negative) Hochschild cohomology $\mathrm{HH}^*_k(R, S)$. Moreover, we study the case

$$\pi_1 \mathrm{Map}_{\mathsf{dgAlg}_k}(R, R)_{id} := [R, R]^{\otimes}_1.$$

Theorem A (cf. 3.5)
Let R in dgAlg_k (a strict DG k-algebra 2.9). There is an exact sequence of groups

$$\cdots \to \mathrm{H}^{-1}(R) \to [R, R]^{\otimes}_1 \to \mathrm{HH}^0_k(R)^\star \to \mathrm{H}^0(R)^\star,$$

where $\mathrm{HH}^0_k(R)^\star$ and $\mathrm{H}^0(R)^\star$ are the groups of units in the rings $\mathrm{HH}^0_k(R)$ and $\mathrm{H}^0(R)$.

Theorem B (cf. 3.8)
Let $\phi : R \to S$ be a morphism in dgAlg_k (such that S is a strict DG k-algebra 2.9). There is an exact sequence of abelian groups

$$\mathrm{H}^{-1}(S) \leftarrow \mathrm{HH}^{-1}_k(R, S) \leftarrow [R, S]^{\otimes}_2 \leftarrow \mathrm{H}^{-2}(S) \leftarrow \mathrm{HH}^{-2}_k(R, S) \leftarrow [R, S]^{\otimes}_3 \leftarrow \dots$$

where S is seen as an R-bimodule via ϕ.

Lemma C (cf 4.2)
If R is a connective DG k-algebra, $R \oplus M$ a connective, square-zero extension, with $\phi : R \to R \oplus M$ the obvious inclusion, then for all $n > 1$

$$\mathrm{Der}^{-n}_k(R, M) \oplus \mathrm{HH}^{-n+1}_k(R) \simeq \pi_n \mathrm{Map}_{\mathsf{dgAlg}_k}(R, R \oplus M)_\phi.$$

2. The six Axioms

We verify the six Axioms described in [4, section 3], for the following categories dgMod_k, dgMod_{R-S} and dgAlg_k. These Axioms will be proved and defined in details (essentially the third Axiom 2.6) the rest are more or less obvious in our setting.

2.1. **Axiom I.** [4, 3.1] The model structures on dgMod_S, dgMod_{R-S} and dgAlg_k are all compatible in the sense that the weak equivalences and fibrations are the underlying weak equivalences and fibrations in dgMod_k. Hence, there is nothing to verify.

2.2. **Axiom II.** [4, 3.2] Let dgMod^0_{R-S} denote the category of pointed DG $R-S$-modules X i.e., coming with a morphism $k \to X$ in dgMod_k. The second axiom requires the existence of a Quilen adjunction

$$\mathsf{dgMod}^0_{R-S} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{dgAlg}_{R-S} \tag{2.1}$$

since all the involved categories are locally presentable [12, proposition 3.7]. The forgetful functor U commutes with limits and directed colimits, therefore the left adjoint exists by [1, p. 65]. The existence of a model structure on dgMod^0_{R-S}

(where weak equivalence and fibrations are underlying weak equivalences and fibrations of dgMod_{R-S}) is guaranteed by [7, Proposition 1.1.8]. Moreover, it is a Quillen adjunction since fibrations and weak equivalences are those of the underlying category dgMod_k, by definition. We denote the image of the unit $1 \in k \to X$ also by $1 \in X$. Now, we give a concrete description of the functor F.

Definition 2.1. The DG $R-S$-algebra $F(X)$ is the quotient of the free DG k-algebra

$$T(X) = k.1 \oplus X \oplus X \otimes X \oplus X^{\otimes^3} \otimes \dots$$

subject to the following relations:

(1) For any $n \in \mathbb{N}^*$, $\underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{\text{n times}} \sim 1$ and the differential of 1 is 0.

(2) For any $s \in S$ and any $x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \cdots \otimes x_n \in T(X)$

$$x_1 \otimes \cdots \otimes x_{i-1} \otimes 1.s \otimes x_{i+1} \cdots \otimes x_n \sim x_1 \otimes \cdots \otimes x_{i-1}.s \otimes x_{i+1} \cdots \otimes x_n.$$

(3) For any $r \in R$ and any $x_1 \otimes \cdots \otimes x_{j-1} \otimes x_{j+1} \cdots \otimes x_m \in T(X)$

$$x_1 \otimes \cdots \otimes x_{j-1} \otimes r.1 \otimes x_{j+1} \cdots \otimes x_m \sim x_1 \otimes \cdots \otimes x_{j-1} \otimes r.x_{j+1} \cdots \otimes x_m.$$

Remark 2.2. The first relation (1) of the previous definition 2.1 is actually redundant.

Recall that the differentials of $T(X)$ are given by (the sign depends on the degree of elements)

$$d(x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \cdots \otimes x_n) = \sum_{i=1}^{n} \pm\, x_1 \otimes \cdots \otimes dx_i \otimes x_{i+1} \cdots \otimes x_n.$$

We can define the following morphisms in dgAlg_k by universal property of $F(X)$

- The morphism $S \to F(X)$ takes s to $1 \otimes 1.s$.
- The morphim $R \to F(X)$ takes r to $r.1 \otimes 1$.
- Notice that in $F(X)$ we have by definition $1 \otimes 1.s = 1.s = 1.s \otimes 1$ for any $s \in S$ and similarly $1 \otimes r.1 = r.1 = r.1 \otimes 1$ for any $r \in R$.

Lemma 2.3. *The functor $F : \mathsf{dgMod}^0_{R-S} \to \mathsf{dgAlg}_{R-S}$ is a left adjoint of the forgetful functor U.*

Proof. In order to prove that F is a left adjoint, we check the universal property i.e., given a morphism $f : X \to U(A)$ in dgMod^0_{R-S} where $A \in \mathsf{dgAlg}_{R-S}$, there is a unique extension $\overline{f} : F(X) \to A$ of DG $R-S$-algebras. By definition, the element $1 \in X$ goes to the unit e of A, such that $f(r.1) = r.e$ and $f(1.s) = e.s$. The equivalence class of the tensor element $x_1 \otimes x_2 \cdots \otimes x_n$ in $F(X)$ is sent by $\overline{f}$ to $f(x_1).f(x_2)\dots f(x_n)$. This morphism is well defined since f is a map of right DG S-modules. Thus, $x_1 \otimes x_j \otimes 1.s \otimes x_{j+2} \dots x_n$ and $x_1 \otimes \dots x_j.s \otimes x_{j+2} \dots x_n$ have the same image. By analogy, elements of the form $x_1 \cdots \otimes x_i \otimes r.1 \otimes x_{i+2} \dots x_n$ and $x_1 \cdots \otimes x_i \otimes r.x_{i+2} \dots x_m$ have the same image by $\overline{f}$. Hence, $\overline{f}$ is uniquely defined. Moreover, any map $\overline{g} : F(X) \to A$ defines, obviously, a unique map of DG $R-S$-bimodules $g : X \to UA$. Therefore, there is an isomorphism of sets

$$\mathsf{dgMod}^0_{R-S}(X, UA) \simeq \mathsf{dgAlg}_{R-S}(F(X), A).$$

□

Remark 2.4. Our construction of the functor F was inspired by a topological analogy. The adjunction between the category of **pointed** topological spaces and the category of topological monoids is given by the forgetful functor and James's functor as left adjoint, we refer to [9].

2.3. **Axiom IV, V and VI.** [4, 3.5, 3.6, 3.8] These axioms are easy to verify. For the fourth Axiom, it is enough to take R cofibrant in dgAlg_k, while the fifth Axiom holds if S is cofibrant in dgMod_S, which is trivial since k is cofibrant in dgMod_k. The last Axiom requires that the two maps defined below are weak equivalences.

- We need $k^c \otimes S \to k \otimes S$ to be an equivalence of right S-modules, where k^c is some cofibrant replacement of k in dgMod_k. This is satisfied, since we can take $k^c = k$.
- For any map $R \to S$ in dgAlg_k, let R^c be a cofibrant replacement of R in the category of DG R-bimodules. The map $R^c \otimes_R S \to R \otimes_R S \simeq S$ is a weak equivalence of $R-S$-modules (in dgMod_k in fact). In order to prove the statement, we use the a concrete model for the cofibrant replacement R^c, which is given by the Bar construction $B(R)$. Since R is cofibrant as DG R-module, then by [5, Proposition 7.5], the natural map $B(R,R,S) \to R \otimes_R S \simeq S$ is a weak equivalence in dgMod_{R-S}. On the other hand, $B(R,R,S)$ is naturally isomorphic to $B(R) \otimes_R S$. We conclude that the morphism $R^c \otimes_R S \to S$ is an equivalence of DG $R-S$-bimodules.

2.4. **Axiom III.**

Definition 2.5. [4, 3.4] A **distinguished object** in dgMod^0_{R-S} is a pointed DG $R-S$-bimodule, such that $k \to X$ induces an equivalence $S \simeq k \otimes S \to X$ of right S-module. An object A in dgAlg_{R-S} is said to be *distinguished* if the map induced by the unit $S \simeq k \otimes S \to A$ is a weak equivalence.

Definition 2.6. [**Axiom III**] We say that the functor $F : \mathsf{dgMod}^0_{R-S} \to \mathsf{dgAlg}_{R-S}$ verifies the third axiom if it sends cofibrant distinguished objects in dgMod^0_{R-S} to (cofibrant) distinguished object in dgAlg_{R-S}.

Lemma 2.7. *Let R be a cofibrant DG k-algebra and M be a cofibrant DG $R-S$-bimodule such that M is zigzag equivalent to S as a right DG S-module. Then there is a map $\phi : R \to S$ of DG-algebras and a weak equivalence $M \to S$ of DG $R-S$-bimodules (where the left action of R on S is induced by ϕ).*

Proof. The proof of this lemma is based on Toën's fundamental theorem [14, Theorem 4.2]. Toën's theorem compares two models for the mapping space of the model category of dg-categories. Since R is a cofibrant DG-algebra, then R is a cofibrant DG-module [14, Proposition 2.3]. By [14, proposition 3.3], a cofibrant DG $R-S$-module M is a cofibrant DG S-module (forgetting the DG R-module structure). It follows by [14, Theorem 4.2] that each cofibrant $R-S$-bimodule M is zigzag equivalent to S where the the left action of R on S is given by some map of DG-algebras $\phi : R \to S$. More precisely, we have the following zigzag of weak equivalences of $R-S$-bimodules

$$S \leftarrow M_1 \leftrightarrow M_2 \cdots \to M,$$

where M_i are cofibrant as DG S-modules. We replace functorially by M_i^c (cofibrant replacement in the category of $R-S$-bimodules, and hence we obtain a weak equivalence (not unique) of DG $R-S$-bimodules $M \to S$. □

Remark 2.8. Under the same hypothesis as in lemma 2.7, if in addition M is pointed (i.e. $k \to M$) then $M \to S$ is an equivalence of pointed DG $R-S$-bimodules.

Definition 2.9. Let S be a DG k-algebra, we say that S is a **strict** if the differential $d_{-1} : S_{-1} \to S_0$ is identically 0.

Definition 2.10. Let S be a DG k-algebra and 1 the unit element, a homotopy invertible element $x \in S_0$ is a cocycle such that there exists an other cocycle $y \in S_0$ with the property that $xy - 1$ and $yx - 1$ are boundaries

Remark 2.11. If S is a strict DG k-algebra, then any homotopy invertible element is strictly invertible.

Lemma 2.12. *Suppose that S is DG algebra where all homotopy invertible elements are strictly invertible. Then any weak equivalence $S \to S$ in dgMod_S is an isomorphism.*

Proof. Any S-linear morphism $f : S \to S$ is determined by the image of the unit 1. Since f is a weak equivalence and and S is fibrant cofibrant in dgMod_S, implies that f has a homotopy inverse g. Hence $fg(1) - 1$ and $gf(1) - 1$ are boundaries. But by hypothesis on S, we have that $fg(1) - 1 = gf(1) - 1 = 0$ □

Remark 2.13. Till the end of the subsection, we will assume that R is a cofibrant DG algbera and S is a strict DG R-algebra 2.9.

Lemma 2.14. *Let S a DG algebra as in 2.13, then the universal map of DG algebras $S \to F(S)$ is an isomorphism.*

Proof. Recall that $R \in \mathsf{dgAlg}_k$ and we have a map of DG algebras $\phi : R \to S$ such that all homotopy invertible elements of S are strictly invertible. Take a representative element in $F(S)$ of the form $s_1 \otimes \cdots \otimes s_n$. Since the chosen element 1 (not the unit in general) of S is strictly invertible, the element $s_1 \otimes \cdots \otimes s_n$ can be reduced to an element of S by using only relations (1) and (2) in 2.1. More precisely

$$s_1 \otimes s_2 \cdots \otimes s_n = s_1 \otimes 1.1^{-1}s_2 \cdots \otimes 1.1^{-1}s_n \sim s_1.1^{-1}s_2 \ldots 1^{-1}s_n = s.$$

Hence, the map $S \to F(S)$ is an isomorphism. □

Definition 2.15. Let $I : \mathsf{dgMod}^0_{R-S} \to \mathsf{dgMod}_{R-S}$ be the functor defined as follows: $I(X)$ is a two sided ideal of $T(X)$ generated by the relations:

(1) $x \otimes 1.s - x.s$ for any element $s \in S$ and any element $x \in X$ where 1 is the image of the unit $k \to X$.
(2) $r.1 \otimes y - r.y$ for any element $r \in R$ and any element $y \in X$

By definition 2.1, it is clear that the quotient in dgMod_{R-S} or dgMod_k of $T(X)$ by $I(X)$ is isomorphic to $F(X)$.

Lemma 2.16. *Let S be a strict DG k-algebra and let $\phi : R \to S$ a map in dgAlg_k which induces a left action of R on S. Let e be the unit of S and fix an invertible element $1 \in S$. Then $\mathrm{H}^*I(X)$ is generated by $1 \otimes 1 - 1$ as $\mathrm{H}^*R \otimes \mathrm{H}^*T(S) - \mathrm{H}^*T(S) \otimes \mathrm{H}^*S$ graded bimodule.*

Proof. By definition $I(S)$ is generated by elements $x \otimes 1.s - x.s$ and $r.1 \otimes y - r.y$ as DG $T(S)$-bimodule 2.15. Since 1 is invertible element, the second kind of generators $r.1 \otimes y - r.y$ can be reduced to the first kind of generators, more precisely

$$r.1 \otimes y - r.y = \phi(r).1 \otimes y - \phi(r).y = \phi(r).1 \otimes 1.1^{-1}y - \phi(r).1.1^{-1}.y$$

which is of the form $x \otimes 1.s - x.s$. Now, we suppose that $\phi : R \to S$ is a fibration i.e. a surjective map, then any generator of the form $x \otimes 1.s - x.s$ can be written as $x.1^{-1}(1 \otimes 1 - 1)s$. In this case $\mathrm{H}^* I(X)$ is generated by $1 \otimes 1 - 1$ as $\mathrm{H}^* R \otimes \mathrm{H}^* T(S) - \mathrm{H}^* T(S) \otimes \mathrm{H}^* S$ graded bimodule. Now, if $\phi : R \to S$ is not a fibration, we factor ϕ as a trivial cofibration followed by a fibration

$$R \xrightarrow[\phi_1]{\sim} R' \xrightarrow[\phi_2]{} S \ ,$$

Since the generators of $I(S)$ are of the form $x \otimes 1.s - x.s$ (do not depend on the action of R on S), we conclude that $\mathrm{H}^* I(S_\phi)$ and $\mathrm{H}^* I(S_{\phi_2})$ for both actions $\phi : R \to S$ or $\phi_2 : R \to S$ are isomorphic. By definition, we have that $\mathrm{H}^* R' \to \mathrm{H}^* R$ is an isomorphism of graded algebras, we conclude that $H^* I(S)$ is generated by $1 \otimes 1 - 1$ as $\mathrm{H}^* R \otimes \mathrm{H}^* T(S) - \mathrm{H}^* T(S) \otimes \mathrm{H}^* S$ graded bimodule for any map of DG algebras $\phi : R \to S$. □

Lemma 2.17. *Let M be a cofibrant distinguished object in dgMod^0_{R-S} (cf 2.5) and $f : M \to S$ be an equivalence in dgMod^0_{R-S} (cf.2.7), then $F(M) \to F(S)$ is an equivalence in dgAlg_{R-S}.*

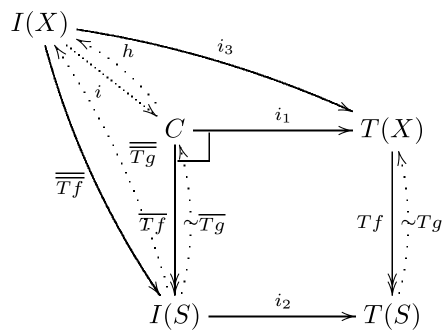
Proof. First of all, since M is cofibrant we can reduce the problem to a trivial fibration of $R - S$-bimodules, namely we factor the pointed weak equivalence in dgMod^0_{R-S} as $M \rightarrowtail X \twoheadrightarrow S$. It is sufficient to prove that $F(X) \to F(S)$ is a weak equivalence. Moreover, the point $\nu : k \to S$ induces an isomorphism of DG right S-modules since S is distinguished (because X is distinguished by assumption) and has the property of the remark 2.13 and 2.9.
The morphism $f : M \to S$ (which is a trivial fibration in dgMod^0_{R-S}) has a section g in the category dgMod^0_S because $k \to S$ induces an isomorphism $k \otimes S \to S$ in dgMod_S since the chosen element in S is strictly invertible, more precisely, we have

$$M \xrightarrow{f} S \xrightarrow[\simeq]{b} S$$

where b is the isomorphism of dgMod_S that takes the base point 1 of S to the unit e of the DGA S (cf 2.12), it follows that $b \circ f$ has a section in dgMod_S taking e to the base point of M, hence f has a section g in dgMod^0_S taking $1 \in S$ to the base point of M, i.e., $f \circ g = id_S$.

This implies that the two sided ideal $I(S)$ is generated only by elements of the form $x \otimes 1.s - x.s$. Hence, the morphism g induces a section $\overline{\overline{Tg}}$ of $\overline{\overline{Tf}} : I(S) \to I(X)$ such that $\overline{\overline{Tg}}\ \overline{\overline{Tf}}$ are restrictions of Tg and Tf.

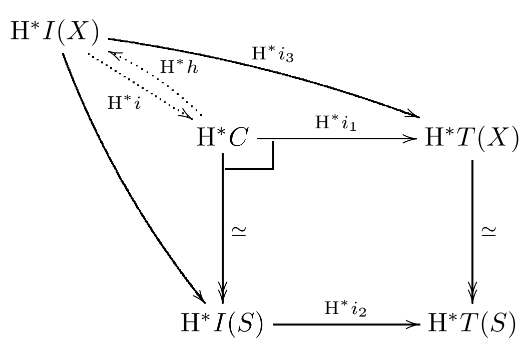
For any trivial fibration of pointed DG $R-S$ module $f : X \twoheadrightarrow S$, we have the following commutative diagram:



satisfying the fallowing relations:

- $Tf \circ Tg = id_{T(S)}$.
- $\overline{Tf} : C \to I(S)$ and $\overline{Tg} : I(S) \to C$ are weak equivalences and $\overline{Tf} \circ \overline{Tg} = id_{I(S)}$.
- $\overline{\overline{\overline{Tf}}} : I(S) \to I(X)$ and $\overline{\overline{\overline{Tg}}} : I(S) \to I(X)$ verify $\overline{\overline{\overline{Tf}}} \circ \overline{\overline{\overline{Tg}}} = id_{I(S)}$.
- $Tf \circ i_1 = i_2 \circ \overline{Tf}$.
- $i_1 \circ i = i_3$.
- $\overline{Tf} \circ i = \overline{\overline{\overline{Tf}}}$.
- Define $h = \overline{\overline{\overline{Tg}}} \circ \overline{Tf} : C \to I(X)$.

Applying the cohomology functor to the previous pullback, we obtain a pullback diagram in the category of pointed $\mathrm{H}^*R \otimes \mathrm{H}^*T(S) - \mathrm{H}^*T(S) \otimes \mathrm{H}^*S$ graded bimodules:



The map $\mathrm{H}^*\overline{\overline{\overline{Tg}}}$ is uniquely defined in the category of pointed $\mathrm{H}^*R \otimes \mathrm{H}^*T(S) - \mathrm{H}^*T(S) \otimes \mathrm{H}^*S$ graded bimodules sending the element (class) $1 \otimes 1 - 1 \in \mathrm{H}^*I(S)$ to $1 \otimes 1 - 1 \in \mathrm{H}^*I(X)$. More precisely, the chosen point in $\mathrm{H}^*T(S)$ is the image of $1 \otimes 1 - 1 \in \mathrm{H}^*I(S)$ by H^*i_2. Obviously, it determines points in H^*C and H^*TX in a canonical way. Notice that $\mathrm{H}^*I(X)$ is already canonically pointed in (cohomolgy class) $1 \otimes 1 - 1$. We deduce that $\mathrm{H}^*h : \mathrm{H}^*C \to \mathrm{H}^*I(X)$ is uniquely defined in the category of pointed graded $\mathrm{H}^*R \otimes \mathrm{H}^*T(S) - \mathrm{H}^*T(S) \otimes \mathrm{H}^*S$-bimodule. In order to prove that $i : I(X) \to C$ is weak equivalence it is sufficient to prove that $\mathrm{H}^*(i_3) \circ \mathrm{H}^*(h) = \mathrm{H}^*(i_1)$. By construction

$$i_1 \circ \overline{Tg} : I(S) \to C \to T(X)$$

and

$$i_3 \circ \overline{\overline{Tg}} : I(S) \to I(X) \to T(X)$$

coincide. Hence,

$$i_1 \circ \overline{Tg} \circ \overline{Tf} = i_3 \circ \overline{\overline{Tg}} \circ \overline{Tf}$$

it implies that

$$i_1 \circ \overline{Tg} \circ \overline{Tf} = i_3 \circ h.$$

On the other hand $\mathrm{H}^*\overline{Tg} \circ \mathrm{H}^*\overline{Tf} = id$, which implies that $\mathrm{H}^*(i_3) \circ \mathrm{H}^*(h) = \mathrm{H}^*(i_1)$. By universality of the pullback, we obtain that $\mathrm{H}^*I(X) \to \mathrm{H}^*C$ is an isomorphism hence $\overline{\overline{Tf}} : I(X) \to I(S)$ is an equivalence. We have the following commutative diagram of short exact sequences in dgMod_k

$$\begin{array}{ccccc} I(X) & \xrightarrow{inc} & T(X) & \twoheadrightarrow & F(X) \\ \downarrow{\sim} & & \downarrow{\sim} & & \downarrow \\ I(S) & \xrightarrow{inc} & T(S) & \twoheadrightarrow & F(S) \end{array}$$

and by five-lemma (exact sequence with five terms), we finally conclude that the map $F(X) \to F(S)$ is a weak equivalence. □

In order to verify the third axiom we have to prove that the derived functor of $F : \mathsf{dgMod}^0_{R-S} \to \mathsf{dgAlg}_{R-S}$ preserves distinguished objects.

Lemma 2.18 (**Axiom III**). *Let R be a cofibrant DG algebra, and S a DG algebra such that all homotopy invertible elements are strictly invertible. Then the left derived functor of $F : \mathsf{dgMod}^0_{R-S} \to \mathsf{dgAlg}_{R-S}$ preserves distinguished objects.*

Proof. It is a consequence of Lemma 2.14 and Lemma 2.17. □

3. The Mapping Space

3.1. **The fundamental group of** $\mathrm{Map}_{\mathsf{dgAlg}_k}(R,R)_{id}$**.** We denote the category of DG categories by $dg-\mathsf{Cat}$, the existence of model structure à la Dwyer-Kan, was initially proved in [13]. In [14], Toën gives a complete description of the mapping space of dg-categories. If R is a cofibrant dg-category and S any dg-category, the mapping space $\mathrm{Map}_{dg-\mathsf{Cat}}(\mathsf{R},\mathsf{S})$ is described as the nerve of the category of *right quasi-representable* $\mathsf{R} \otimes \mathsf{S}^{op}$-dg-functors [14, Definition 4.1]. In the particular case where R and S have just one object with the underlying DG algebras R and S , then the right quasi-representable $\mathsf{R} \otimes \mathsf{S}^{op}$-dg-functors, are exactly, the $R-S$-bimodules which are (zig-zag) equivalent to S as right S-module. In [4, p.15], such $R-S$-bimodules are called *potentially distinguished* modules. The nerve of the category of potentially distinguished modules is denoted by $\mathcal{M}_{R,S}$ and if $R = k$ we denote it simply by $\mathcal{M}_S$. Notice that if R is cofibrant in dgAlg_k then R is cofibrant in $dg-\mathsf{Cat}$. We summarize the previous discussion in the following Lemma.

Lemma 3.1. *The moduli space $\mathcal{M}_{R,S}$ is equivalent to $\mathrm{Map}_{dg-\mathsf{Cat}}(\mathsf{R},\mathsf{S})$ and $\mathcal{M}_S$ is equivalent to $\mathrm{Map}_{dg-\mathsf{Cat}}(\mathsf{k},\mathsf{S})$, where R (resp. S and k) is the dg-categorie with one object an the underlying endomorphim DG algebra R (resp. S and k).*

Proof. It is a direct consequence of the computation of the mapping space in the model category of DG categories ($dg-\mathsf{Cat}$) cf. [14, Theorem 4.2] and [4, section 1.13]. □

Remark 3.2. [**Base points**] If $\phi : R \to S$ is a morphism of DG algebras, the moduli space $\mathcal{M}_{R,S}$ is pointed at the object S which is a canonical distinguished object equivalent to S as a right S-module and has a structure of $R-S$-bimodule via ϕ. The corresponding base point of $\mathrm{Map}_{dg-\mathsf{Cat}}(\mathsf{R},\mathsf{S})$ is the morphism ϕ. By the same way, the moduli space $\mathcal{M}_S$ is also pointed at the object S, and the corresponding base point of $\mathrm{Map}_{dg-\mathsf{Cat}}(\mathsf{k},\mathsf{S})$ is the unit morphism $k \to S$. Finally, the base point of the space $\mathrm{Map}_{\mathsf{dgAlg}_k}(R,S)$ is the morphism ϕ.

Now, we are ready to introduce the Dwyer-Hess fundamental Theorems [4, Theorem 3.10 and 3.11] in the context of dgAlg_k.

Theorem 3.3. *Let $\phi : R \to S$ be a morphism of DG algebras (such that S is a strict DG k-algebra 2.9) . There exists a fiber sequence*

$$\mathrm{Map}_{\mathsf{dgAlg}_k}(R,S) \to \mathcal{M}_{R,S} \to \mathcal{M}_S,$$

or equivalently, the fiber sequence:

$$\mathrm{Map}_{\mathsf{dgAlg}_k}(R,S) \to \mathrm{Map}_{dg-\mathsf{Cat}}(\mathsf{R},\mathsf{S}) \to \mathrm{Map}_{dg-\mathsf{Cat}}(\mathsf{k},\mathsf{S}).$$

Proof. Since the categories dgMod_k, dgAlg_k and dgMod_{R-S} verify the six Axioms (2) for any cofibrant DG k-algebra R and any strict DG k-algebra S. Hence, there is a fiber sequence $\mathrm{Map}_{\mathsf{dgAlg}_k}(R,S) \to \mathcal{M}_{R,S} \to \mathcal{M}_S$ by [4, Theorem 3.10]. For the second fiber sequence, we apply Lemma 3.1. □

Remark 3.4. Notice that the homotopy limits in dgAlg_k and $dg-\mathsf{Cat}^*$ (where $dg-\mathsf{Cat}^*$ is the full subcategory of $dg-\mathsf{Cat}$ such that all categories have only one object) are the same and the fact that the mapping space commutes with homotopy limits, we have a more general result:

$$\mathrm{Map}_{\mathsf{dgAlg}_k}(R,S) \to \mathrm{Map}_{dg-\mathsf{Cat}}(\mathsf{R},\mathsf{S}) \to \mathrm{Map}_{dg-\mathsf{Cat}}(\mathsf{k},\mathsf{S}).$$

is a fiber sequence for any cofibrant R and any DG k-algebra S which is homotopy limit of strcit DG k-algebras. Unfortunately, we don't know if any DG k-algebra is a homotopy limit of strict DG k-algebras.

Theorem 3.5. *Let R be a strict DG k-algebra, then there is an exact sequence of groups*

$$\cdots \to \mathrm{H}^{-1}(R) \to [R,R]_1^{\otimes} \to \mathrm{HH}_k^0(R)^\star \to \mathrm{H}^0(R)^\star.$$

where $\mathrm{HH}_k^0(R)^\star$ is the group of units of $\mathrm{HH}_k^0(R)$ and $\mathrm{H}^0(R)^\star$ is the group of the units of the ring $\mathrm{H}^0(R)$.

Proof. By Theorem 3.3, we have the fiber sequence

$$\mathrm{Map}_{\mathsf{dgAlg}_k}(R,R)_{id} \to \mathrm{Map}_{dg-\mathsf{Cat}}(\mathsf{R},\mathsf{R})_{id} \to \mathrm{Map}_{dg-\mathsf{Cat}}(\mathsf{k},\mathsf{R})_{\nu}.$$

Therefore, applying the long exact sequence of homotopy groups (starting at level one), we obtain

$$\pi_1\mathrm{Map}_{dg-\mathsf{Cat}}(\mathsf{k},\mathsf{R})_{\nu} \leftarrow \pi_1\mathrm{Map}_{dg-\mathsf{Cat}}(\mathsf{R},\mathsf{R})_{id} \leftarrow [R,R]_1^{\otimes} \leftarrow \pi_2\mathrm{Map}_{dg-\mathsf{Cat}}(\mathsf{k},\mathsf{R})_{id}\dots$$

By [14, Corollary 8.3], we have that

$$\pi_1\mathrm{Map}_{dg-\mathsf{Cat}}(\mathsf{R},\mathsf{R})_{id} \simeq \mathrm{HH}_k^0(R)^\star.$$

By [14, corollary 4.10], we have that for $i > 0$.

$$\pi_1\mathrm{Map}_{dg-\mathsf{Cat}}(k,\mathsf{R})_{\nu} \simeq \mathrm{H}^0(R)^\star \text{ and } \pi_{i+1}\mathrm{Map}_{dg-\mathsf{Cat}}(k,\mathsf{R})_{\nu} \simeq \mathrm{H}^{-i}(R).$$

□

Corollary 3.6. *Let R be a DG algebra such that $\mathrm{H}^{-1}(R) = 0$, then*

$$\pi_1 \mathrm{Map}_{\mathsf{dgAlg}_k}(R,R)_{id} \simeq Ker[\mathrm{HH}^0_k(R)^\star \to \mathrm{H}^0(R)^\star].$$

3.2. **Higher Homotopy Groups of** $\mathrm{Map}_{\mathsf{dgAlg}_k}(R,S)_\phi$**.** In this section, we give a complete description of the higher homotopy groups of the mapping space of the model category dgAlg_k.

Theorem 3.7. *Let $\nu : k \to S$ be the unit, and $\phi : R \to S$ a map of dgAlg_k and S is a strict DG k-algebra. There is a fiber sequence of spaces:*

$$\Omega \mathrm{Map}_{\mathsf{dgAlg}_k}(R,S)_\phi \to \mathrm{Map}_{\mathsf{dgMod}_{R-R}}(R, S_\phi) \to \mathrm{Map}_{\mathsf{dgMod}_k}(k,S),$$

where S_ϕ is seen as R-bimodule via ϕ.

Proof. It is a consequence of 3.3 and [4, Theorem 3.11]. □

Theorem 3.8. *For any map of DG algebras $\phi : R \to S$ such that S is a strict DG k-algebra, there is a long exact sequence of abelian groups*

$$\mathrm{H}^{-1}(S) \leftarrow \mathrm{HH}^{-1}_k(R,S) \leftarrow [R,S]^\otimes_2 \leftarrow \mathrm{H}^{-2}(S) \leftarrow \mathrm{HH}^{-2}_k(R,S) \leftarrow [R,S]^\otimes_3 \leftarrow \dots$$

where S is seen as an R-bimodule via ϕ.

Proof. Let $\nu : k \to S$ be the unit morphism. We loop the previous fiber sequence 3.7 and obtain a new fiber sequence

$$\Omega^2 \mathrm{Map}_{\mathsf{dgAlg}_k}(R,S)_\phi \to \Omega\mathrm{Map}_{\mathsf{dgMod}_{R-R}}(R, S_\phi) \to \Omega\mathrm{Map}_{\mathsf{dgMod}_k}(k,S). \tag{3.1}$$

Since the model categories dgMod_{R-R} and dgMod_k are stable and in particular pointed, the map

$$\Omega\mathrm{Map}_{\mathsf{dgMod}_{R-R}}(R, S_\phi) \to \Omega\mathrm{Map}_{\mathsf{dgMod}_k}(k,S)$$

induces a morphism of abelian groups

$$\pi_i\Omega\mathrm{Map}_{\mathsf{dgMod}_{R-R}}(R, S_\phi) \to \pi_i\Omega\mathrm{Map}_{\mathsf{dgMod}_k}(k,S) \text{ for } i \geq 0.$$

By 0.2 and Definition 0.4, there is an isomorphism of groups

$$\pi_i\Omega\mathrm{Map}_{\mathsf{dgMod}_{R-R}}(R, S_\phi) \simeq \mathrm{HH}^{-1-i}_k(R,S) \text{ for } i \geq 0,$$

and by 0.6 $\pi_i\Omega\mathrm{Map}_{\mathsf{dgMod}_k}(k,S) \simeq \mathrm{H}^{-i-1}(S)$. Therefore, applying the Serre exact sequence to 3.1 we prove our theorem.

□

Corollary 3.9. *If $\phi : R \to S$ a morphism of DG k-algebras such that S is connective, then*

$$\mathrm{HH}^{-i}_k(R,S) \simeq [R,S]^\otimes_{i+1} \text{ for all } i > 0.$$

Proof. Since S is connective, i.e., $\mathrm{H}^n(S) = 0$ for all strictly negative integers. According to the long exact sequence in 3.8, $[R,S]^\otimes_{i+1} \simeq \mathrm{HH}^{-i}_k(R,S)$ for all $i > 0$. □

4. Applications

In all our application we assume that S is a strict Dg k-algebra.

4.1. **Infinity loop space.** The first evident consequence of 3.7 is the extra structure on the double loop space of $\mathrm{Map}_{\mathsf{dgAlg}_k}(R,S)$ which is summarized in the following result.

Corollary 4.1. *Let $\phi : R \to S$ a map of DG algebras, such that S is connective, then $\Omega^2_\phi \mathrm{Map}_{\mathsf{dgAlg}_k}(R,S)$ is an infinity loop space.*

Proof. Since $\pi_i \mathrm{Map}_{\mathsf{dgMod}_k}(k,S)$ vanishes for $i>0$ and $\mathrm{Map}_{\mathsf{dgMod}_{R-R}}(R,S_\phi)$ is an infinity loop space because dgMod_{R-R} is a stable model category. Hence, by Theorem 3.7, we conclude that

$$\Omega \mathrm{Map}_{\mathsf{dgMod}_{R-R}}(R,S_\phi) \sim \Omega^2_\phi \mathrm{Map}_{\mathsf{dgAlg}_k}(R,S)$$

is an infinity loop space. □

4.2. **Derivations.** We make a connection with the theory of derivations of DG k-algebras. Let R be in dgAlg_k and M a DG R-bimodule. We define a new DG R-algebra $R\oplus M$, called a *square zero extension* as follows. It is the DG algebras whose underlying complex is $R\oplus M$ and whose DG algebra structure is the obvious one induced from the trivial multiplication on M, i.e., $m.m^{'}=0$ for any $m,m^{'}\in M$. The map $\phi : R \to R\oplus M$ is the obvious map of DG R-algebras. In [3], the authors use the inverse gradation, i.e., **the differentials are of degree -1**. According to the long exact sequence described in [3, 3.14] and their notations, if M is coconnective then,

$$\mathrm{Der}^{-n}_k(R,M) \simeq \mathrm{HH}^{-n+1}_k(R,M) \text{ for } n>1. \tag{4.1}$$

But with our gradation and notation if M is connective then $\mathrm{Der}^{-n}(R,M)\simeq \mathrm{HH}^{-n+1}_k(R,M)$ for all $n>1$, and we have the following lemma

Lemma 4.2. *If R is a connective DG k-algebra, $R\oplus M$ is a connective square-zero extension graded differential R-bimodule, and $\phi : R\to R\oplus M$ the obvious inclusion, then for all $n>1$*

$$\mathrm{Der}^{-n}_k(R,M)\oplus \mathrm{HH}^{-n+1}_k(R,R) \simeq \pi_n \mathrm{Map}_{\mathsf{dgAlg}_k}(R,R\oplus M)_\phi.$$

Proof. It is a consequence of 3.9, 4.1, and the fact that the Hochschild cohomology is additive. □

4.3. **Commutative DG algebras.** Let $k=\mathbb{Q}$ or any field of characteristic 0. The model category of commutative differential unbounded k-algebras is denoted by dgCAlg_k equipped with the induced model structure, i.e., weak equivalences are isomorphisms in homology (Cf. [15, section 2.3.1]) and fibrations are degree-wise surjective morphisms. There is a Quillen adjunction

$$\mathsf{dgAlg}_k \underset{U}{\overset{Ab}{\rightleftarrows}} \mathsf{dgCAlg}_k \ ,$$

where Ab is called the abelianization functor. If $R\in \mathsf{dgAlg}_k$ is cofibrant and $S\in \mathsf{dgCAlg}_k$, then by [4, Theorem 2.12], there is a weak homotopy equivalence of simplicial sets

$$\mathrm{Map}_{\mathsf{dgCAlg}_k}(Ab(R),S) \sim \mathrm{Map}_{\mathsf{dgAlg}_k}(R,S). \tag{4.2}$$

Let $S\in \mathsf{dgCAlg}_k$, and let $\phi : R\to S$ be a morphism of DG algebras, then it induces a morphism $\overline{\phi} : Ab(R)\to S$ in dgCAlg_k.

Corollary 4.3. *Let R be a cofibrant DG algebra, and let S be a connective commutative DG algebra with $\phi : R \to S$ is a map of DG k-algebras. Then, there is an isomorphism of abelian groups*

$$\pi_i \mathrm{Map}_{\mathsf{dgCAlg}_k}(Ab(R), S)_{\overline{\phi}} \simeq \mathrm{HH}_k^{1-i}(R, S), \ i > 1.$$

Proof. It is a formal consequence of 3.9 and the fact that $\mathrm{Map}_{\mathsf{dgCAlg}_k}(Ab(R), S)_{\overline{\phi}} \sim \mathrm{Map}_{\mathsf{dgAlg}_k}(R, S)_\phi$ by 4.2. □

Remark 4.4. By the Eckmann-Hilton argument, the category dgCAlg_k is the category of monoids in the monoidal category $(\mathsf{dgAlg}_k, \otimes)$ (Cf.[10, Section 4]). It is tempting to apply Dwyer-Hess fundamental theorem for dgCAlg_k in order to compute $\mathrm{Map}_{\mathsf{dgCAlg}_k}(R, S)$ for any commutative DG algebras R and S. The problem is the **Axiom III** (Cf. 2.4), which is not verified in general for dgCAlg_k, otherwise it would mean that for any commutative DG algebra R, the natural map $Ab(R^c) \to R$ is a weak equivalence in dgCAlg_k (where, R^c is a cofibrant replacement of R in dgAlg_k). An easy example, due to Lurie, is the free commutative algebra in two variables $R = \mathbb{Q}[x, y]$. The cofibrant replacement of R is the free associative DG algebra R^c in three variables $x, y, \ z$ such that $\deg(x) = \deg(y) = 0$ and $dz = xy - yx$. if S is any commutative DG algebra, by simple computation, we obtain $\pi_0 \mathrm{Map}_{\mathsf{dgCAlg}_k}(R, S) \simeq \mathrm{H}^0(S) \oplus \mathrm{H}^0(S)$, but $\pi_0 \mathrm{Map}_{\mathsf{dgAlg}_k}(R^c, S) \simeq \mathrm{H}^0(S) \oplus \mathrm{H}^0(S) \oplus \mathrm{H}^{-1}(S)$. We conclude that $Ab(R^c) \to R$ is not an equivalence in general.

4.4. **Conclusion.** It is natural to ask the following questions:
Question 1: Are Theorems 3.5 and 3.8 still true if we replace k by any commutative DG algebra A?
Question 2: What is the correct formulation of Theorems 3.5 and 3.8 in the setting of the stable monoidal model category of symmetric spectra Sp and the associated category of ring spactra $\mathsf{Sp}^{\otimes}$?

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References

[1] J. Adámek and J. Rosickỳ. *Locally presentable and accessible categories*, volume 189. Cambridge Univ Pr, 1994.

[2] I. Amrani. Model structure on the category of topological categories. *arxiv.org/pdf/1110.2695*, 2011.

[3] D. Dugger and B. Shipley. Topological equivalences for differential graded algebras. *Advances in Mathematics*, 212(1), 2007.

[4] W. Dwyer and K. Hess. Long knots and maps between operads. *Geometry & Topology*, 16:919–955, 2012.

[5] A.D. Elmendorf, I. Kriz, M.A. Mandell, J.P. May, and M. Cole. *Rings, modules, and algebras in stable homotopy theory*. American Mathematical Society, 2007.

[6] S. I. Gelfand and Y. Manin. *Methods of homological algebra: Springer monographs in mathematics*. Springer, 2003.

[7] M. Hovey. Model categories. *Mathematical Surveys and Monographs*, 63, 1999.

[8] Henning Krause. Derived categories, resolutions, and brown representability. *Contemporary Mathematics*, 436, 2007.

[9] Vogt R. M. Algebras and modules in monoidal model categories. *Homotopy homomorphisms and the classifying space functor*, http://arxiv.org/pdf/1203.4978.pdf, 2012.

[10] Hans-E Porst. On categories of monoids, comonoids, and bimonoids. *Quaestiones Mathematicae*, 31(2), 2008.

[11] S. Schwede and B.E. Shipley. Algebras and modules in monoidal model categories. *Proceedings of the London Mathematical Society*, 80(2):491, 2000.

[12] B. Shipley. HZ-algebra spectra are differential graded algebras. *American journal of mathematics*, 129(2), 2007.

[13] G. Tabuada. Une structure de catégorie de modeles de Quillen sur la catégorie des dg-catégories. *Comptes Rendus Mathematique*, 340(1):15–19, 2005.

[14] B. Toën. The homotopy theory of dg-categories and derived Morita theory. *Inventiones Mathematicae*, 167(3):615–667, 2007.

[15] B. Toën and G. Vezzosi. *Homotopical algebraic geometry II: Geometric stacks and applications*. Amer Mathematical Society, 2008.

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