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# Holonomy algebras of pseudo-hyper-Kählerian manifolds of index 4

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ABSTRACT

The holonomy algebra of a pseudo-hyper-Kählerian manifold of signature $(4, 4n+4)$ is a subalgebra of $\mathfrak{sp}(1, n+1)$. Possible holonomy algebras of these manifolds are classified. Using this, a new proof of the classification of simply connected pseudo-hyper-Kählerian symmetric spaces of index 4 is obtained.



## 1. Introduction

The classification of holonomy algebras of Riemannian manifolds is well known and it has a lot of applications both in geometry and physics, see e.g. [5,6,10,18,20]. Lately the theory of pseudo-Riemannian geometries has been steadily developing. In particular, a classification of holonomy algebras of pseudo-Riemannian manifolds is an actual problem of differential geometry. It is solved only in some cases. The difficulty appears if the holonomy algebra preserves a degenerate subspace of the tangent space. Classification of holonomy algebras of Lorentzian manifolds is obtained in [24,3,23,12,14]; classification of holonomy algebras of pseudo-Kählerian manifolds of index 2 is achieved in [13]. These algebras are contained in $\mathfrak{so}(1, n+1)$ and $\mathfrak{u}(1, n+1) \subset \mathfrak{so}(2, 2n+2)$, respectively. There are partial results for holonomy algebras of pseudo-Riemannian manifolds of signature $(2, n)$ and $(n, n)$ [19,4,17]. More details can be found in the recent review [15].

In [9] holonomy algebras of pseudo-quaternionic-Kählerian manifolds with non-zero scalar curvature are classified. These algebras $\mathfrak{g}$ are contained in $\mathfrak{sp}(1) \oplus \mathfrak{sp}(r, s)$ and they contain $\mathfrak{sp}(1)$. If $s \neq r$, then $\mathfrak{g}$ is irreducible. If $s = r$, then $\mathfrak{g}$ may preserve a degenerate subspace of the tangent space, in this case there are only two possibilities for $\mathfrak{g}$. This strong result follows mainly from the inclusion $\mathfrak{sp}(1) \subset \mathfrak{g}$.

Recall that a pseudo-hyper-Kählerian manifold is a pseudo-Riemannian manifold $(M, h)$ together with three parallel $g$-orthogonal complex structures $I_1$, $I_2$, $I_3$ that satisfy the relations $I_1^2 = I_2^2 = I_3^2 = -\mathrm{id}$, $I_3 = I_1 I_2 = -I_2 I_1$. Any such manifold has signature $(4r, 4s)$, $r + s > 1$, and its holonomy algebra $\mathfrak{g}$ is contained in $\mathfrak{sp}(r, s)$. Conversely, any simply connected pseudo-Riemannian manifold with such holonomy algebra is pseudo-hyper-Kählerian. Note that any pseudo-hyper-Kählerian manifold is also pseudo-quaternionic-Kählerian and it has zero scalar curvature.

In the present paper we classify all possible holonomy algebras $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)$ of pseudo-hyper-Kählerian manifolds of signature $(4, 4n+4)$, $n \geqslant 1$. For $n = 0$ this classification is obtained in [8]. The main results is stated in Section 3. Section 4 is dedicated to their proofs. To prove the classification theorem we use the fact that a holonomy algebra $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)$ is

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a Berger algebra, i.e. $\mathfrak{g}$ is spanned by the images of the algebraic curvature tensors $R \in \mathcal{R}(\mathfrak{g})$ of type $\mathfrak{g}$. Recall that $\mathcal{R}(\mathfrak{g})$ is the space of linear maps from $\bigwedge^2 \mathbb{R}^{4,4n+4}$ to $\mathfrak{g}$ satisfying the first Bianchi identity. In [7] weakly irreducible subalgebras $\mathfrak{g} \subset \mathfrak{sp}(1,n+1)$ containing a certain ideal $\operatorname{Im}\mathbb{H}$ (see decomposition (3)) are partially classified. Here we find missing subalgebras, then we compute the spaces $\mathcal{R}(\mathfrak{g})$ for each of these algebras and we check which $\mathfrak{g}$ are Berger algebras. Then we show that each weakly-irreducible Berger subalgebra $\mathfrak{g} \subset \mathfrak{sp}(1,n+1)$ contains $\operatorname{Im}\mathbb{H}$. This gives the classification of weakly-irreducible not irreducible Berger subalgebras $\mathfrak{g} \subset \mathfrak{sp}(1,n+1)$. Remark that in this paper only possible holonomy algebras are listed. We do not know if all these algebras may appear as the holonomy algebras, to show this examples of manifolds must be constructed. Since the most of the previously known Berger algebras are realized as the holonomy algebras, one may expect that the algebras obtained here can be realized as the holonomy algebras of pseudo-hyper-Kählerian manifolds.

In [1,21,22] simply connected pseudo-hyper-Kählerian symmetric spaces of index 4 are classified. Using the results of this paper we give a new simple proof of this classification. We use the known fact that a simply connected pseudo-Riemannian symmetric space is uniquely defined (up to a homothety) by its holonomy algebra $\mathfrak{g}$ and an algebraic curvature tensor $R \in \mathcal{R}(\mathfrak{g})$ such that the representation of $\mathfrak{g}$ in $\mathcal{R}(\mathfrak{g})$ annihilates $R$ and the image of $R$ spans $\mathfrak{g}$. In Section 5 we describe all such pairs $(\mathfrak{g}, R)$ for $\mathfrak{g} \subset \mathfrak{sp}(1,n+1)$. We show that if a pseudo-hyper-Kählerian manifold $(M,h)$ of signature $(4,4n+4)$, $n \geqslant 1$ is locally symmetric, then $n=2$ and we give explicitly the curvature tensor and holonomy algebra of the obtained space. For the case of signature $(4,4)$ the analogous result is obtained in [8].

## 2. Preliminaries

Let $\mathbb{H}^m$ be an $m$-dimensional quaternionic vector space. A pseudo-quaternionic-Hermitian metric $g$ on $\mathbb{H}^m$ is a non-degenerate $\mathbb{R}$-bilinear map $g:\mathbb{H}^m \times \mathbb{H}^m \to \mathbb{H}$ such that $g(aX,Y) = ag(X,Y)$ and $\overline{g(Y,X)} = g(X,Y)$, where $a \in \mathbb{H}$, $X, Y \in \mathbb{H}^m$. Hence, $g(X,aY) = g(X,Y)\bar{a}$. There exists a basis $e_1,\dots,e_m$ of $\mathbb{H}^m$ and integers $(r,s)$ with $r+s=m$ such that $g(e_t,e_l)=0$ if $t \neq l$, $g(e_t,e_t) = -1$ if $1 \leqslant t \leqslant r$ and $g(e_t,e_t)=1$ if $r+1 \leqslant t \leqslant m$. The pair $(r,s)$ is called the signature of $g$. In this situation we denote $\mathbb{H}^m$ by $\mathbb{H}^{r,s}$. The realification of $\mathbb{H}^m$ gives us the vector space $\mathbb{R}^{4m}$ with the quaternionic structure $(i,j,k)$. Conversely, a quaternionic structure on $\mathbb{R}^{4m}$, i.e. a triple $(I_1,I_2,I_3)$ of endomorphisms of $\mathbb{R}^{4m}$ such that $I_1^2 = I_2^2 = I_3^2 = -\operatorname{id}$ and $I_3 = I_1 I_2 = -I_2 I_1$, allows us to consider $\mathbb{R}^{4m}$ as $\mathbb{H}^m$. A pseudo-quaternionic-Hermitian metric $g$ on $\mathbb{H}^m$ of signature $(r,s)$ defines on $\mathbb{R}^{4m}$ the $i$, $j$, $k$-invariant pseudo-Euclidean metric $\eta$ of signature $(4r,4s)$, $\eta(X,Y) = \operatorname{Re} g(X,Y)$, $X,Y \in \mathbb{R}^{4m}$. Conversely, a $I_1$, $I_2$, $I_3$-invariant pseudo-Euclidean metric on $\mathbb{R}^{4m}$ defines a pseudo-quaternionic-Hermitian metric $g$ on $\mathbb{H}^m$,

$$g(X,Y) = \eta(X,Y) + i\eta(X,I_1Y) + j\eta(X,I_2Y) + k\eta(X,I_3Y).$$

We will identify $(1,i,j,k)$ with $(I_0,I_1,I_2,I_3)$, respectively. The identification $\mathbb{R}^{4r,4s} \simeq \mathbb{H}^{r,s}$ allows to multiply the vectors of $\mathbb{R}^{4r,4s}$ by quaternionic numbers.

The Lie algebra $\mathfrak{sp}(r,s)$ is defined as follows

$$\begin{aligned}\mathfrak{sp}(r,s) &= \bigl\{ f \in \mathfrak{so}(4r,4s) \bigm| [f,I_1] = [f,I_2] = [f,I_3] = 0 \bigr\} \\ &= \bigl\{ f \in \operatorname{End}\bigl(\mathbb{H}^{r,s}\bigr) \bigm| g(fX,Y) + g(X,fY) = 0 \text{ for all } X,Y \in \mathbb{H}^{r,s} \bigr\}.\end{aligned}$$

Denote by $\mathfrak{sp}(1)$ the subalgebra in $\mathfrak{so}(4r,4s)$ generated by the $\mathbb{R}$-linear maps $I_1$, $I_2$, $I_3$.

Clearly, the tangent space of a pseudo-hyper-Kählerian manifold $(M,h)$ at a point $x \in M$ one can identify with $(\mathbb{R}^{4r,4s},\eta,I_1,I_2,I_3) = (\mathbb{H}^{r,s},g)$. Then the holonomy algebra of a pseudo-hyper-Kählerian manifold is identified with a subalgebra $\mathfrak{g} \subset \mathfrak{sp}(r,s)$.

Let $(V,\eta)$ be a pseudo-Euclidean space and $\mathfrak{g} \subset \mathfrak{so}(V)$ be a subalgebra. The space of curvature tensors $\mathcal{R}(\mathfrak{g})$ of type $\mathfrak{g}$ is defined as follows

$$\mathcal{R}(\mathfrak{g}) = \left\{ R \in \operatorname{Hom}\left( \bigwedge^2 V, \mathfrak{g} \right) \,\middle|\, R(u,v)w + R(v,w)u + R(w,u)v = 0 \text{ for all } u,v,w \in V \right\}.$$

Denote by $L(\mathcal{R}(\mathfrak{g}))$ the vector subspace of $\mathfrak{g}$ spanned by the elements $R(u,v)$ for all $R \in \mathcal{R}(\mathfrak{g})$ and $u,v \in V$. *A subalgebra $\mathfrak{g} \subset \mathfrak{so}(r,s)$ is called a Berger algebra if $L(\mathcal{R}(\mathfrak{g})) = \mathfrak{g}$.* From the Ambrose–Singer Theorem it follows that if $\mathfrak{g} \subset \mathfrak{so}(V)$ is the holonomy algebra of a pseudo-Riemannian manifold, then $\mathfrak{g}$ is a Berger algebra. Therefore, Berger algebras may be considered as the candidates to the holonomy algebras.

Now we summarize some facts about quaternionic vector spaces. Let $\mathbb{H}^m$ be an $m$-dimensional quaternionic vector space and $e_1,\dots,e_m$ a basis of $\mathbb{H}^m$. We identify an element $X \in \mathbb{H}^m$ with the column $(X_t)$ of the left coordinates of $X$ with respect to this basis, $X = \sum_{t=1}^m X_t e_t$. Let $f:\mathbb{H}^m \to \mathbb{H}^m$ be an $\mathbb{H}$-linear map. Define the matrix $\operatorname{Mat}_f$ of $f$ by the relation $fe_l = \sum_{t=1}^m (\operatorname{Mat}_f)_{tl} e_t$. Now if $X \in \mathbb{H}^m$, then $fX = (X^t \operatorname{Mat}_f^t)^t$ and because of the non-commutativity of the quaternionic numbers this is not the same as $\operatorname{Mat}_f X$. Conversely, to an $m \times m$ matrix $A$ of the quaternionic numbers we put in correspondence the linear map $\operatorname{Op} A:\mathbb{H}^m \to \mathbb{H}^m$ such that $\operatorname{Op} A \cdot X = (X^t A^t)^t$. If $f,g:\mathbb{H}^m \to \mathbb{H}^m$ are two $\mathbb{H}$-linear maps, then $\operatorname{Mat}_{fg} = (\operatorname{Mat}_g^t \operatorname{Mat}_f^t)^t$. Note that the multiplications by the imaginary quaternionic numbers are not $\mathbb{H}$-linear maps. Also, for $a,b \in \mathbb{H}$ holds $\overline{ab} = \bar{b}\bar{a}$. Consequently, for two square quaternionic matrices we have $(\overline{AB})^t = \bar{B}^t \bar{A}^t$.

## 3. Results

Let $(M,h)$ be a pseudo-hyper-Kählerian manifold of signature $(4,4n+4)$, $n \geqslant 1$. The tangent space to the manifold $(M,h)$ at a point $x \in M$ can be identified with the pseudo-Euclidean space $(\mathbb{R}^{4,4n+4}, \eta, I_1, I_2, I_3)$, where $\eta$ is a pseudo-Euclidean metric on $\mathbb{R}^{4,4n+4}$, $(I_1, I_2, I_3)$ is the quaternionic structure on $\mathbb{R}^{4,4n+4}$. This space one can identify with the pseudo-quaternionic-Hermitian space $(\mathbb{H}^{1,n+1}, g)$, where $g$ is a pseudo-quaternionic-Hermitian metric on $\mathbb{H}^{1,n+1}$.

The Wu Theorem [26] allows to assume that the manifold $(M,h)$ is locally indecomposable, i.e. locally it is not a product of pseudo-Riemannian manifolds of positive dimensions. This happens if and only if the holonomy algebra $\mathfrak{g} \subset \mathfrak{sp}(1,n+1)$ of $(M,h)$ does not preserve any proper non-degenerate subspace of $\mathbb{R}^{4,4n+4}$. Such subalgebras $\mathfrak{g} \subset \mathfrak{sp}(1,n+1)$ are called *weakly irreducible*. If the holonomy algebra $\mathfrak{g} \subset \mathfrak{sp}(1,n+1)$ is irreducible, then $\mathfrak{g} = \mathfrak{sp}(1,n+1)$ [5,10,25]. Thus we may assume that $\mathfrak{g} \subset \mathfrak{sp}(1,n+1)$ is weakly irreducible and not irreducible. In this case $\mathfrak{g}$ preserves a four-dimensional isotropic $I_1$, $I_2$, $I_3$-invariant subspace $W \subset \mathbb{H}^{1,n+1}$. We fix a non-zero vector $p \in W$, then $W = \mathbb{H}p$. Let $q \in \mathbb{H}^{1,n+1}$ be any isotropic vector such that $g(p,q) = 1$. Denote by $\mathbb{H}^n$ the $g$-orthogonal complement to $\mathbb{H}p \oplus \mathbb{H}q$ in $\mathbb{H}^{1,n+1}$. Let $e_1, \dots, e_n$ be a basis of $\mathbb{H}^n$ and let $G$ be the corresponding Gram matrix of $g|_{\mathbb{H}^n}$, i.e. $G_{ab} = g(e_a, e_b)$. Denote by $\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$ the maximal subalgebra of $\mathfrak{sp}(1,n+1)$ that preserves the quaternionic isotropic line $\mathbb{H}p$, this Lie algebra has the matrix form:

$$\mathfrak{sp}(1,n+1)_{\mathbb{H}p} = \left\{ \mathrm{Op}\begin{pmatrix} a & -(G\bar{X})^t & b \\ 0 & \mathrm{Mat}_A & X \\ 0 & 0 & -\bar{a} \end{pmatrix} \middle| \, a \in \mathbb{H},\ A \in \mathfrak{sp}(n),\ X \in \mathbb{H}^n,\ b \in \mathrm{Im}\,\mathbb{H} \right\}. \tag{1}$$

Here $\mathrm{Op}\,\mathcal{M}$ denotes the $\mathbb{H}$-linear endomorphism of $\mathbb{H}^{1,n+1}$ given by a matrix $\mathcal{M}$, see Section 2.

Notice that to find the matrix form of $\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$ acting in $\mathbb{R}^{4,4n+4}$ one can consider the basis $p, I_1p, I_2p, I_3p, e_1, \dots, I_3e_n, q, I_1q, I_2q, I_3q$ of $\mathbb{R}^{4,4n+4}$ and it is enough to change each element $c = c_0 + c_1 i + c_2 j + c_3 k$ of the matrix from (1) to the matrix

$$\begin{pmatrix} c_0 & -c_1 & -c_2 & -c_3 \\ c_1 & c_0 & c_3 & -c_2 \\ c_2 & -c_3 & c_0 & c_1 \\ c_3 & c_2 & -c_1 & c_0 \end{pmatrix}.$$

We denote the element from (1) by the quadruple $(a, A, X, b)$. One can easily find the following Lie brackets:

$$\big[(a,0,0,0), \big(a',0,X,b\big)\big] = \big(a'a - aa', 0, \bar{a}X, 2\,\mathrm{Im}\,ba\big),$$
$$\big[(0,0,X,0),(0,0,Y,0)\big] = \big(0,0,0,2\,\mathrm{Im}\,g(X,Y)\big),$$
$$\big[(0,A,0,0),(0,B,X,0)\big] = \big(0,[A,B]_{\mathfrak{sp}(n)}, AX, 0\big),$$

where $a, a' \in \mathbb{H}$, $X, Y \in \mathbb{H}^n$, $A, B \in \mathfrak{sp}(n)$, $b \in \mathrm{Im}\,\mathbb{H}$.

Thus we get the decomposition

$$\mathfrak{sp}(1,n+1)_{\mathbb{H}p} = \mathbb{H} \oplus \mathfrak{sp}(n) \ltimes \big(\mathbb{H}^n \ltimes \mathrm{Im}\,\mathbb{H}\big). \tag{2}$$

We may also write $\mathbb{H} = \mathbb{R} \oplus \mathfrak{sp}(1)$, then

$$\mathfrak{sp}(1,n+1)_{\mathbb{H}p} = \mathbb{R} \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \ltimes \big(\mathbb{H}^n \ltimes \mathrm{Im}\,\mathbb{H}\big). \tag{3}$$

The isomorphism $\{(a,0,0,0) \mid a \in \mathrm{Im}\,\mathbb{H}\} \simeq \mathfrak{sp}(1)$ is given by $(a,0,0,0) \mapsto -a$.

The ideal $\mathbb{H}^n \ltimes \mathrm{Im}\,\mathbb{H} \subset \mathfrak{sp}(1,n+1)_{\mathbb{H}p}$ is isomorphic to the quaternionic Heisenberg Lie algebra. The Levi–Malcev decomposition of $\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$ has the form

$$\mathfrak{sp}(1,n+1)_{\mathbb{H}p} = \mathfrak{s} \ltimes \mathfrak{r}, \quad \mathfrak{s} = \mathfrak{sp}(1) \oplus \mathfrak{sp}(n),\ \mathfrak{r} = \mathbb{R} \ltimes \big(\mathbb{H}^n \ltimes \mathrm{Im}\,\mathbb{H}\big),$$

where $\mathfrak{s}$ is a semisimple subalgebra and $\mathfrak{r}$ is the radical of $\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$. We may write the $\mathbb{Z}$-grading

$$\mathfrak{sp}(1,n+1)_{\mathbb{H}p} = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2, \quad \mathfrak{g}_0 = \mathbb{R} \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(n),\ \mathfrak{g}_1 = \mathbb{H}^n,\ \mathfrak{g}_2 = \mathrm{Im}\,\mathbb{H}$$

with the grading element $1 \in \mathbb{R} \subset \mathfrak{sp}(1,n+1)_{\mathbb{H}p}$, i.e. $\mathrm{ad}_1|_{\mathfrak{g}_\alpha} = \alpha\,\mathrm{id}_{\mathfrak{g}_\alpha}$, $\alpha = 0,1,2$.

Let $m$, $m_1$, $m_2$ be integers such that either $m + m_1 + m_2 = n$ or $m + m_1 + m_2 \leqslant n - 2$. Set the following denotation:

$$\mathbb{H}^m = \mathrm{span}_{\mathbb{H}}\{e_1, \dots, e_m\},$$
$$\mathrm{Im}\,\mathbb{H}^{m_1} = i\mathbb{R}^{m_1} \oplus j\mathbb{R}^{m_1} \oplus k\mathbb{R}^{m_1}, \quad \text{where } \mathbb{R}^{m_1} = \mathrm{span}_{\mathbb{R}}\{e_{m+1}, \dots, e_{m+m_1}\},$$
$$\mathbb{C}^{m_2} = \mathrm{span}_{\mathbb{R} \oplus i\mathbb{R}}\{e_{m+m_1+1}, \dots, e_{m+m_1+m_2}\}.$$

Let $L'$ be a real vector subspace of $\mathrm{span}_{\mathbb{H}}\{e_{m+m_1+m_2+1}, \dots, e_n\}$ coinciding with a $g$-orthogonal direct sum of the real spaces of the form

$$\mathrm{span}_{\mathbb{R}}\{f_1,\dots,f_l, if_1+jf_2,\dots,if_{l-1}+jf_l\}, \quad l \geqslant 2,$$

where we fix a fragmentation of the interval $[m+m_1+m_2+1,\dots,n]$ of natural numbers into a disjunct union of subintervals of length at least 2 and $f_1,\dots,f_l$ are vectors from the set $\{e_{m+m_1+m_2+1},\dots,e_n\}$ corresponding to one of these subintervals.

Consider the following real vector subspace of $\mathbb{H}^n$:

$$L = L\big(m,m_1,m_2,L'\big) = \mathbb{H}^m \oplus \mathrm{Im}\,\mathbb{H}^{m_1} \oplus \mathbb{C}^{m_2} \oplus L'. \tag{4}$$

Assume that the decomposition (4) is $g$-orthogonal. Let $g$ be defined by this and the following conditions:

1) $g_{ab} = \delta_{ab}$, if $1 \leqslant a,b \leqslant m$;
2) $g_{ab} = \delta_{ab} + iw_{1ab} + jw_{2ab} + kw_{3ab}$, if $m+1 \leqslant a,b \leqslant m+m_1$, where $w_1, w_2, w_3$ are skew-symmetric bilinear forms on $\mathbb{R}^{m_1}$;
3) $g_{ab} = \delta_{ab} + w_{ab}j$, if $m+m_1+1 \leqslant a,b \leqslant m+m_1+m_2$, where $w$ is a skew-symmetric $\mathbb{C}$-bilinear form on $\mathbb{C}^{m_2}$;
4) $g_{ab} = \eta_{ab} + i\Omega_{1ab} + j\Omega_{2ab} + k\Omega_{3ab}$, if $m+m_1+m_2+1 \leqslant a,b \leqslant n$, where $\eta_{ab}$ is a positive definite symmetric bilinear form on $\mathrm{span}_{\mathbb{R}}\{e_{m+m_1+m_2+1},\dots,e_n\}$ and $\Omega_1, \Omega_2, \Omega_3$ are skew-symmetric bilinear forms on $\mathrm{span}_{\mathbb{R}}\{e_{m+m_1+m_2+1},\dots,e_n\}$.

The above forms may be degenerate or zero.

Recall that any subalgebra $\mathfrak{h} \subset \mathfrak{sp}(n)$ can be decomposed as $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{z}(\mathfrak{h})$, where $\mathfrak{h}' = [\mathfrak{h},\mathfrak{h}]$ is the commutant of $\mathfrak{h}$ and $\mathfrak{z}(\mathfrak{h})$ is the center of $\mathfrak{h}$.

Now we state the classification result. In the first theorem we provide a general description of possible holonomy algebras. In the second theorem we give the precise list of all possible holonomy algebras. It will be enough to prove only the second theorem.

**Theorem 1.** *Let $(M,h)$ be a locally indecomposable pseudo-hyper-Kählerian manifold of signature $(4,4n+4)$, $n \geqslant 1$. If the holonomy algebra $\mathfrak{g}$ of $(M,h)$ is not irreducible, then $\mathfrak{g}$ is conjugated by an element of $\mathrm{SO}(4,4n+4)$ to one of the following subalgebras of $\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$:*

I. $\mathfrak{g}_I = \bar{\mathfrak{h}} \ltimes (\mathbb{H}^m \oplus \mathbb{C}^{n-m} \ltimes \mathrm{Im}\,\mathbb{H})$, *where* $0 \leqslant m \leqslant n$,
*if* $m = n$, *then* $\bar{\mathfrak{h}} \subset \mathbb{R} \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ *is a subalgebra such that* $\dim \mathrm{pr}_{\mathbb{H}} \bar{\mathfrak{h}} \neq 1$;
*if* $m < n$, *then* $\bar{\mathfrak{h}} \subset \mathbb{R} \oplus i\mathbb{R} \oplus \mathfrak{sp}(m)$ *is a subalgebra such that* $\dim \mathrm{pr}_{\mathbb{R}\oplus i\mathbb{R}} \bar{\mathfrak{h}} \neq 1$.

II. $\mathfrak{g}_{II} = \mathfrak{h} \ltimes (L(m,m_1,m_2,L') \ltimes \mathrm{Im}\,\mathbb{H})$, *where* $\mathfrak{h} \subset \mathfrak{sp}(m)$ *is a subalgebra.*

III. $\mathfrak{g}_{III} = \bar{\mathfrak{h}} \ltimes (\mathbb{H}^m \oplus \mathrm{Im}\,\mathbb{H}^{n-m} \ltimes \mathrm{Im}\,\mathbb{H})$, *where* $0 \leqslant m < n$,
$\bar{\mathfrak{h}} \subset \{a + \mathrm{Op}(aE_{n-m}) \mid a \in \mathfrak{sp}(1)\} \oplus \mathfrak{sp}(m)$
*and* $\mathrm{pr}_{\{a+\mathrm{Op}(aE_{n-m})\mid a\in\mathfrak{sp}(1)\}} \bar{\mathfrak{h}} = \{a + \mathrm{Op}(aE_{n-m}) \mid a \in \mathfrak{sp}(1)\}$,
*here* $\mathrm{Op}(aE_{n-m}) \in \mathfrak{sp}(n-m)$ *is the element with the matrix* $aE_{n-m}$.

IV. $\mathfrak{g}_{IV} = \{A + \psi(A) \mid A \in \mathfrak{h}\} \ltimes (\mathbb{H}^k \oplus V \ltimes \mathrm{Im}\,\mathbb{H})$,
*here* $L(m,m_1,m_2,L') = \mathbb{H}^k \oplus V \oplus U$ *is an* $\eta$*-orthogonal decomposition* ($\eta = \mathrm{Re}\,g$), $\mathfrak{h} \subset \mathfrak{sp}(k)$ *is a subalgebra,* $\psi : \mathfrak{h} \to U$ *is a surjective linear map and* $\psi|_{\mathfrak{h}'} = 0$.

V. $\mathfrak{g}_V = \{A + \psi(A) \mid A \in \bar{\mathfrak{h}}\} \ltimes (\mathbb{H}^{m_0} \oplus \mathrm{Im}\,\mathbb{H}^{n-m_0} \ltimes \mathrm{Im}\,\mathbb{H})$,
*here* $0 < m_0 < m \leqslant n$, $\bar{\mathfrak{h}} \subset \{a + \mathrm{Op}(aE_{n-m_0}) \mid a \in \mathfrak{sp}(1)\} \oplus \mathfrak{sp}(m_0)$ *is a subalgebra with* $\mathrm{pr}_{\{a+\mathrm{Op}(aE_{n-m_0})\mid a\in\mathfrak{sp}(1)\}} \bar{\mathfrak{h}} = \{a + \mathrm{Op}(aE_{n-m_0}) \mid a \in \mathfrak{sp}(1)\}$, $\psi : \bar{\mathfrak{h}} \to \mathrm{span}_{\mathbb{R}}\{e_{m_0+1},\dots,e_m\}$ *is a surjective linear map with* $\psi|_{\bar{\mathfrak{h}}'} = 0$.

**Theorem 2.** *Let $(M,h)$ be a locally indecomposable pseudo-hyper-Kählerian manifold of signature $(4,4n+4)$, $n \geqslant 1$. If the holonomy algebra $\mathfrak{g}$ of $(M,h)$ is not irreducible, then $\mathfrak{g}$ is conjugated by an element of $\mathrm{SO}(4,4n+4)$ to one of the following subalgebras of $\mathfrak{sp}(1,n+1)_{\mathbb{H}p}$:*

I.1. $\mathfrak{g}_{I1} = \mathbb{R} \oplus \mathfrak{h}_0 \oplus \mathfrak{h} \ltimes (\mathbb{H}^m \oplus \mathbb{C}^{n-m} \ltimes \mathrm{Im}\,\mathbb{H})$, *where* $0 \leqslant m \leqslant n$, $\mathfrak{h}_0 \subset \mathfrak{sp}(1)$, $\mathfrak{h} \subset \mathfrak{sp}(m)$ *are subalgebras,*
$\mathfrak{h}_0 = \mathbb{R}i$ *or* $\mathfrak{h}_0 = \mathfrak{sp}(1)$. *If* $m < n$, *then* $\mathfrak{h}_0 = \mathbb{R}i$.

I.2. $\mathfrak{g}_{I2} = \mathbb{R} \oplus \{\phi(A) + A \mid A \in \mathfrak{h}\} \ltimes (\mathbb{H}^m \oplus \mathbb{C}^{n-m} \ltimes \mathrm{Im}\,\mathbb{H})$,
*where* $1 \leqslant m \leqslant n$, $\mathfrak{h} \subset \mathfrak{sp}(m)$ *is a subalgebra,* $\phi : \mathfrak{h} \to \mathfrak{sp}(1)$ *is a non-zero homomorphism.*
*If* $m < n$, *then* $\mathrm{Im}\,\phi = \mathbb{R}i$, $\phi|_{\mathfrak{h}'} = 0$. *If* $m = n$, *then either* $\mathrm{Im}\,\phi = \mathbb{R}i$ *and* $\phi|_{\mathfrak{h}'} = 0$, *or* $\mathrm{Im}\,\phi = \mathfrak{sp}(1)$.

I.3. $\mathfrak{g}_{I3} = \mathfrak{h}_0 \oplus \{\varphi(A) + A \mid A \in \mathfrak{h}\} \ltimes (\mathbb{H}^m \oplus \mathbb{C}^{n-m} \ltimes \mathrm{Im}\,\mathbb{H})$,
*where* $0 \leqslant m \leqslant n$, $\mathfrak{h}_0 \subset \mathfrak{sp}(1)$, $\mathfrak{h} \subset \mathfrak{sp}(m)$ *are subalgebras,* $\varphi : \mathfrak{h} \to \mathbb{R}$ *is a linear map,* $\varphi|_{\mathfrak{h}'} = 0$.
*If* $m < n$, *then* $\mathfrak{h}_0 = \mathbb{R}i$ *and* $\varphi \neq 0$. *If* $m = n$, *then either* $\mathfrak{h}_0 = \mathbb{R}i$ *and* $\varphi \neq 0$, *or* $\mathfrak{h}_0 = \mathfrak{sp}(1)$.

I.4. $\mathfrak{g}_{I4} = \{\varphi(A) + \phi(A) + A \mid A \in \mathfrak{h}\} \ltimes (\mathbb{H}^m \oplus \mathbb{C}^{n-m} \ltimes \mathrm{Im}\,\mathbb{H})$,
*where* $0 \leqslant m \leqslant n$, $\mathfrak{h} \subset \mathfrak{sp}(m)$ *is a subalgebra,* $\varphi : \mathfrak{h} \to \mathbb{R}$, $\phi : \mathfrak{h} \to \mathfrak{sp}(1)$ *are homomorphisms.*
*If* $m < n$, *then either* $\varphi = \phi = 0$ *or* $\varphi \neq 0$, $\mathrm{Im}\,\phi = \mathbb{R}i$ *and the maps* $i\varphi, \phi : \mathfrak{h} \to \mathbb{R}i$ *are not proportional,* $\varphi|_{\mathfrak{h}'} = \phi|_{\mathfrak{h}'} = 0$.

I.5. $\mathfrak{g}_{I5} = \mathbb{R}(\alpha + i) \oplus \{\varphi(A) + A \mid A \in \mathfrak{h}\} \ltimes (\mathbb{H}^m \oplus \mathbb{C}^{n-m} \ltimes \mathrm{Im}\,\mathbb{H})$, *where* $0 \leqslant m \leqslant n$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\mathfrak{h} \subset \mathfrak{sp}(m)$ *is a subalgebra,* $\varphi : \mathfrak{h} \to \mathbb{R}$ *is a non-zero linear map with* $\varphi|_{\mathfrak{h}'} = 0$.

II. $\mathfrak{g}_{II} = \mathfrak{h} \ltimes (L(m,m_1,m_2,L') \ltimes \mathrm{Im}\,\mathbb{H})$, *where* $\mathfrak{h} \subset \mathfrak{sp}(m)$ *is a subalgebra.*

III.1. $\mathfrak{g}_{III1} = \{a + \mathrm{Op}(aE_{n-m}) \mid a \in \mathfrak{sp}(1)\} \oplus \mathfrak{h} \ltimes (\mathbb{H}^m \oplus \mathrm{Im}\,\mathbb{H}^{n-m} \ltimes \mathrm{Im}\,\mathbb{H})$, *where* $n - m \geqslant 1$, $\mathfrak{h} \subset \mathfrak{sp}(m)$ *is a subalgebra, and* $\mathrm{Op}(aE_{n-m}) \in \mathfrak{sp}(n-m)$ *is the element with the matrix* $aE_{n-m}$.

III.2. $\mathfrak{g}_{III2} = \{\phi(A) + A + \mathrm{Op}(\phi(A)E_{n-m}) \mid A \in \mathfrak{h}\} \ltimes (\mathbb{H}^m \oplus \mathrm{Im}\,\mathbb{H}^{n-m} \ltimes \mathrm{Im}\,\mathbb{H})$, *where* $n - m \geqslant 1$, $\mathfrak{h} \subset \mathfrak{sp}(m)$ *is a subalgebra, and* $\phi : \mathfrak{h} \to \mathfrak{sp}(1)$ *is a surjective homomorphism.*

IV. $\mathfrak{g}_{IV} = \{A + \psi(A) \mid A \in \mathfrak{h}\} \ltimes (\mathbb{H}^k \oplus V \ltimes \mathrm{Im}\,\mathbb{H})$, *here* $L(m, m_1, m_2, L') = \mathbb{H}^k \oplus V \oplus U$ *is an* $\eta$*-orthogonal decomposition* ($\eta = \mathrm{Re}\, g$), $\mathfrak{h} \subset \mathfrak{sp}(k)$ *is a subalgebra,* $\psi : \mathfrak{h} \to U$ *is a surjective linear map and* $\psi|_{\mathfrak{h}'} = 0$.

V.1. $\mathfrak{g}_{V1} = \{a + \mathrm{Op}(aE_{n-m_0}) \mid a \in \mathfrak{sp}(1)\} \oplus \{A + \psi(A) \mid A \in \mathfrak{h}\} \ltimes (\mathbb{H}^{m_0} \oplus \mathrm{Im}\,\mathbb{H}^{n-m_0} \ltimes \mathrm{Im}\,\mathbb{H})$, *here* $0 < m_0 < m \leqslant n$, $\mathfrak{h} \subset \mathfrak{sp}(m_0)$ *is a subalgebra,* $\psi : \mathfrak{h} \to \mathrm{span}_{\mathbb{R}}\{e_{m_0+1}, \dots, e_m\}$ *is a surjective linear map with* $\psi|_{\mathfrak{h}'} = 0$.

V.2. $\mathfrak{g}_{V2} = \{a + \chi(a) + \mathrm{Op}(aE_{n-m_0}) \mid a \in \mathfrak{sp}(1)\} \oplus \{A + \psi(A) \mid A \in \mathfrak{h}\} \ltimes (\mathbb{H}^{m_0} \oplus \mathrm{Im}\,\mathbb{H}^{n-m_0} \ltimes \mathrm{Im}\,\mathbb{H})$, *where the dates are the same as for* $\mathfrak{g}_{V1}$ *and in addition* $\chi : \mathfrak{sp}(1) \to \mathfrak{sp}(m_0)$ *is an injective homomorphism such that* $\chi(\mathfrak{sp}(1))$ *commutes with* $\mathfrak{h}$.

*Conversely, all these algebras are Berger algebras.*

*Another description of* $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$. For convenience we give another description of the Lie algebra $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$. One usually identifies the Lie algebra $\mathfrak{so}(r, s)$ with the space of bivectors $\Lambda^2\mathbb{R}^{r,s}$ in such a way that $(X \wedge Y)Z = \eta(Z, X)Y - \eta(Z, Y)X$. Having the pseudo-quaternionic-Hermitian metric $g$ on $\mathbb{H}^{1,n+1}$ we may put

$$(X \wedge_g Y)Z = g(Z, X)Y - g(Z, Y)X,$$

then

$$X \wedge_g Y = \sum_{s=0}^{3} I_r X \wedge I_s Y, \qquad (aX) \wedge_g Y = X \wedge_g (\bar{a}Y),$$

and $X \wedge_g Y \in \mathfrak{sp}(1, n+1)$. We get the identification

$$\mathfrak{sp}(1, n+1) \simeq \Lambda^2_g \mathbb{H}^{1,n+1} = \{X \wedge_g Y \mid X, Y \in \mathbb{H}^{1,n+1}\}.$$

The element $(a, A, X, b) \in \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ corresponds to

$$-(ap) \wedge_g q + A + p \wedge_g X + p \wedge_g \left(\frac{b}{2}p\right), \quad A \in \mathfrak{sp}(n) \simeq \Lambda^2_g \mathbb{H}^n.$$

Let us rewrite Proposition 4.1 given below in these notations.

**Proposition 3.1.** *Any* $R \in \mathcal{R}(\mathfrak{sp}(1, n+1)_{\mathbb{H}p})$ *is uniquely defined by elements* $C_1, C_2 \in \mathbb{H}$, $A_1, A_2, A_3 \in \mathfrak{sp}(n)$, $S_1, S_2 \in \mathbb{H}^n$, $R' \in \mathcal{R}(\mathfrak{sp}(n))$, $P \in \mathcal{P}(\mathfrak{sp}(n))$, $d_1, \dots, d_5 \in \mathbb{R}$ *in the following way*:

$$R(I_s p, q) = p \wedge_g \left(\frac{1}{2}B_s p\right), \qquad R(q, X) = P(X) + p \wedge_g T(X) + p \wedge_g \left(\frac{1}{2}\theta(X)p\right),$$

$$R(X, Y) = R'(X, Y) + p \wedge_g \big(P(Y)X - P(X)Y\big) + p \wedge_g \left(\frac{1}{2}\big(g\big(Y, T(X)\big) - g\big(X, T(Y)\big)\big)p\right),$$

$$R(q, I_s q) = -(C_s p) \wedge_g q + A_s + p \wedge_g S_s + p \wedge_g \left(\frac{1}{2}D_s q\right),$$

$$R(p, I_s p) = R(p, X) = 0, \quad X, Y \in \mathbb{H}^n,$$

*where*

$$C_3 = C_2 i - C_1 j, \qquad T = -\frac{1}{2}(I_1 A_1 + I_2 A_2 + I_3 A_3), \qquad S_3 = jS_1 - iS_2,$$

$$D_1 = d_1 i + d_2 j + d_3 k, \qquad D_2 = d_2 i + d_4 j + d_5 k, \qquad D_3 = jD_1 - iD_2,$$

$$B_s = \frac{1}{2}(I_1 I_s C_1 + I_2 I_s C_2 + I_3 I_s C_3), \qquad \theta(X) = \frac{1}{2}\big(I_1 g(X, S_1) + I_2 g(X, S_2) + I_3 g(X, S_3)\big).$$

The other values of $R$ can be found using equality (6) given below. Although this version of Proposition 4.1 is not so complicated, the form of Proposition 4.1 is more convenient for the proof of Theorem 2.

## 4. Proof of Theorem 2

Since $\mathfrak{g}$ is weakly irreducible and not irreducible, $\mathfrak{g}$ preserves a degenerate vector subspace $V \subset \mathbb{R}^{4,4n+4}$. Let $V_1 = V \cap V^{\perp}$, then $V_1$ is isotropic and $\dim V_1 \leqslant 4$. Let $V_2 = V_1^{\perp} \cap I_1 V_1^{\perp}$. Clearly, $V_2 \neq 0$ and it is degenerate, $\mathfrak{g}$-invariant and $I_1$-invariant. Then $V_3 = V_2 \cap V_2^{\perp}$ is isotropic, $\mathfrak{g}$-invariant and $I_1$-invariant. Starting with $V_3$ in the same way it can be shown that $\mathfrak{g}$ preserves an isotropic $I_1$, $I_2$-invariant subspace $W \subset \mathbb{R}^{4,4n+4}$, then $W$ is also $I_3$-invariant and it has dimension 4. Consequently, $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$.

The proof of the theorem will consist of several parts.

### 4.1. The structure of the space $\mathcal{R}(\mathfrak{sp}(1, n+1)_{\mathbb{H}p})$

Let us find the space of curvature tensors $\mathcal{R}(\mathfrak{sp}(1, n+1)_{\mathbb{H}p})$ for the Lie algebra $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$.

Using the form $\eta$, the Lie algebra $\mathfrak{so}(4, 4n+4)$ can be identified with the space

$$\bigwedge^2 \mathbb{R}^{4,4n+4} = \operatorname{span}\{u \wedge v = u \otimes v - v \otimes u \mid u, v \in \mathbb{R}^{4,4n+4}\}$$

in such a way that $(u \wedge v)w = \eta(u, w)v - \eta(v, w)u$ for all $u, v, w \in \mathbb{R}^{4,4n+4}$. One can check that the element $\operatorname{Op}\begin{pmatrix} a & -(G\bar{X})^t & b \\ 0 & \mathrm{Mat}_A & X \\ 0 & 0 & -\bar{a} \end{pmatrix} \in \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ corresponds to the bivector

$$\begin{aligned}
&\big(-a_0(p \wedge q + ip \wedge iq + jp \wedge jq + kp \wedge kq) + a_1(p \wedge iq - ip \wedge q + kp \wedge jq - jp \wedge kq)\\
&\quad + a_2(-jp \wedge q + p \wedge jq - kp \wedge iq + ip \wedge kq) + a_3(-kp \wedge q + jp \wedge iq - ip \wedge jq + p \wedge kq)\big)\\
&\quad + A + \big((p \wedge X_0 + ip \wedge iX_0 + jp \wedge jX_0 + kp \wedge kX_0) + (p \wedge iX_1 - ip \wedge X_1 - jp \wedge kX_1 + kp \wedge jX_1)\\
&\quad + (p \wedge jX_2 + ip \wedge kX_2 - jp \wedge X_2 - kp \wedge iX_2) + (p \wedge kX_3 - ip \wedge jX_3 + jp \wedge iX_3 - kp \wedge X_3)\big)\\
&\quad + \big(b_1(p \wedge ip - jp \wedge kp) + b_2(p \wedge jp + ip \wedge kp) + b_3(p \wedge kp - ip \wedge jp)\big),
\end{aligned}$$

where $X = X_0 + iX_1 + jX_2 + kX_3$, $X_0, \dots, X_3 \in \mathbb{R}^n = \operatorname{span}_{\mathbb{R}}\{e_1, \dots, e_n\} \subset \mathbb{R}^{4n} \simeq \mathbb{H}^n$, $A \in \mathfrak{sp}(n) \subset \mathfrak{so}(4n) \simeq \bigwedge^2 \mathbb{R}^{4n}$.

Let $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$, $R \in \mathcal{R}(\mathfrak{g})$. It is known that any $R \in \mathcal{R}(\mathfrak{g})$ satisfies

$$\eta\big(R(u, v)z, w\big) = \eta\big(R(z, w)u, v\big) \tag{5}$$

for all $u, v, w, z \in \mathbb{R}^{4,4n+4}$. Using this property, we get the following:

$$\eta\big(R(I_\alpha u, v)z, w\big) = \eta\big(R(z, w)I_\alpha u, v\big) = \eta\big(I_\alpha R(z, w)u, v\big) = -\eta\big(R(z, w)u, I_\alpha v\big) = -\eta\big(R(u, I_\alpha v)z, w\big),$$

i.e. for any $1 \leqslant \alpha \leqslant 3$ and $u, v \in \mathbb{R}^{4,4n+4}$,

$$R(I_\alpha u, v) = -R(u, I_\alpha v) \tag{6}$$

holds. Hence,

$$R(xu, v) = R(u, \bar{x}v) \tag{7}$$

for all $x \in \mathbb{H}$ and $u, v \in \mathbb{R}^{4,4n+4}$.

The metric $\eta$ defines the metric $\eta \wedge \eta$ on $\bigwedge^2 \mathbb{R}^{4,4n+4}$. Using the above identification, $R$ can be considered as the map $R : \bigwedge^2 \mathbb{R}^{4,4n+4} \to \mathfrak{g} \subset \mathfrak{so}(4, 4n+4) \simeq \bigwedge^2 \mathbb{R}^{4,4n+4}$. From (5), we obtain

$$\eta \wedge \eta\big(R(u \wedge v), z \wedge w\big) = \eta \wedge \eta\big(R(z \wedge w), u \wedge v\big) \tag{8}$$

for all $u, v, z, w \in \mathbb{R}^{4,4n+4}$. This shows that $R$ is a symmetric linear map. Consequently $R$ is zero on the orthogonal complement to $\mathfrak{g}$ in $\bigwedge^2 \mathbb{R}^{4,4n+4}$. In particular, the vectors $q \wedge iq + jq \wedge kq$, $q \wedge jq - iq \wedge kq$, $q \wedge kq + iq \wedge jq$, $I_r p \wedge I_s p$, $I_r p \wedge X$, where $X \in \mathbb{R}^{4n}$, are contained in the orthogonal complement to $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$. Hence,

$$R(q, iq) = -R(jq, kq), \qquad R(q, jq) = R(iq, kq), \qquad R(q, kq) = -R(iq, jq), \qquad R(I_r p, I_s p) = R(I_r p, X) = 0. \tag{9}$$

For a subalgebra $\mathfrak{h} \subset \mathfrak{so}(m)$ define the space

$$\mathcal{P}(\mathfrak{h}) = \big\{P \in (\mathbb{R}^m)^* \otimes \mathfrak{h} \mid \eta\big(P(x)y, z\big) + \eta\big(P(y)z, x\big) + \eta\big(P(z)x, y\big) = 0 \text{ for all } x, y, z \in \mathbb{R}^m\big\},$$

where $\eta$ is the scalar product on $\mathbb{R}^m$. This space is studied in [23,16].

**Proposition 4.1.** *Any $R \in \mathcal{R}(\mathfrak{sp}(1,n+1)_{\mathbb{H}p})$, $n \geqslant 1$ is uniquely defined by elements $C_{01}, C_{02} \in \mathbb{H}$, $A_{01}, A_{02}, A_{03} \in \mathfrak{sp}(n)$, $S_{01}, S_{02} \in \mathbb{H}^n$, $R' \in \mathcal{R}(\mathfrak{sp}(n))$, $P_0 \in \mathcal{P}(\mathfrak{sp}(n))$, $d_1, \dots, d_5 \in \mathbb{R}$ in the following way:*

$$R(I_r p, I_s q) = (0,0,0,B_{rs}), \qquad R(I_s q, X) = \big(0, P_s(X), T_s(X), \theta_s(X)\big),$$
$$R(X,Y) = \big(0, R'(X,Y), Q(X,Y), \tau(X,Y)\big), \qquad R(I_r q, I_s q) = (C_{rs}, A_{rs}, S_{rs}, D_{rs}),$$
$$R(I_r p, I_s p) = R(I_r p, X) = 0, \quad X, Y \in \mathbb{R}^{4n},$$

*where*

$$C_{03} = C_{02} i - C_{01} j, \qquad C_{12} = -C_{03}, \qquad C_{13} = C_{02}, \qquad C_{23} = -C_{01}, \qquad C_{rs} = C_{0r} I_s - C_{0s} I_r, \quad r, s \neq 0, \tag{10}$$
$$D_{01} = d_1 i + d_2 j + d_3 k, \qquad D_{02} = d_2 i + d_4 j + d_5 k, \qquad D_{03} = j D_{01} - i D_{02}, \tag{11}$$
$$D_{23} = -D_{01}, \qquad D_{13} = D_{02}, \qquad D_{12} = -D_{03}, \qquad D_{rs} = I_r D_{0s} - I_s D_{0r}, \tag{12}$$
$$A_{23} = -A_{01}, \qquad A_{13} = A_{02}, \qquad A_{12} = -A_{03}, \tag{13}$$
$$T_0 = -\frac{1}{2}(I_1 A_{01} + I_2 A_{02} + I_3 A_{03}), \qquad T_s = I_s T_0 - A_{0s} = -T_0 I_s, \quad s \neq 0, \tag{14}$$
$$P_s = -P_0 \circ I_s, \quad s \neq 0, \qquad Q(X,Y) = P_0(Y)X - P_0(X)Y, \tag{15}$$
$$B_{rs} = I_r C_{0s} + I_s B_{r0}, \quad B_{r0} = \frac{1}{2}(I_1 I_r C_{01} + I_2 I_r C_{02} + I_3 I_r C_{03}), \tag{16}$$
$$\tau(X,Y) = g\big(Y, T_0(X)\big) - g\big(X, T_0(Y)\big), \tag{17}$$
$$S_{03} = j S_{01} - i S_{02}, \qquad S_{23} = -S_{01}, \qquad S_{13} = S_{02}, \qquad S_{12} = -S_{03}, \qquad S_{rs} = I_r S_{0s} - I_s S_{0r}, \tag{18}$$
$$\theta_0(X) = \frac{1}{2}\big(I_1 g(X, S_{01}) + I_2 g(X, S_{02}) + I_3 g(X, S_{03})\big), \tag{19}$$
$$\theta_s(X) = g(X, S_{0s}) + I_s \theta_0(X) = -\theta_0(I_s X), \quad s \neq 0, \tag{20}$$

*where $X, Y \in \mathbb{H}^n$. Moreover, it holds*

$$\eta\big(Q(Y,Z), X\big) = \eta\big(P_0(X)Y, Z\big), \qquad \eta\big(I_r \tau(X,Y)p, I_s q\big) = \eta(A_{rs}X, Y),$$
$$\eta\big(I_r \theta_s(x)p, I_t q\big) = \eta(I_s S_{rt}, X), \qquad \eta(I_t B_{rs} p, I_{t_1} q) = \eta(I_r C_{tt_1} p, I_s q), \tag{21}$$

*where $X, Y, Z \in \mathbb{R}^{4n}$.*

**Proof.** Let $R \in \mathcal{R}(\mathfrak{sp}(1,n+1)_{\mathbb{H}p})$.

The equality (9) shows that $R(I_r p, I_s p) = R(I_r p, X) = 0$. We may write

$$R(I_r p, I_s q) = (\lambda_{rs}, F_{rs}, X_{rs}, B_{rs}), \qquad R(I_s q, X) = \big(\mu_s(X), P_s(X), T_s(X), \theta_s(X)\big),$$
$$R(X,Y) = \big(\sigma(X,Y), R'(X,Y), Q(X,Y), \tau(X,Y)\big), \qquad R(I_r q, I_s q) = (C_{rs}, A_{rs}, S_{rs}, D_{rs}).$$

Now we find the conditions that satisfy the obtained elements. By the Bianchi identity,

$$R(I_r p, I_s q)X + R(I_s q, X)I_r p + R(X, I_r p)I_s q = 0.$$

Using the equality $R(X, I_r p) = 0$ and taking the projection on $\mathbb{H}^n$, we get $F_{rs}X = 0$, i.e. $F_{rs} = 0$. Using (5), we obtain

$$\eta\big(R(X,Y)I_r p, I_s q\big) = \eta\big(R(I_r p, I_s q)X, Y\big) = 0,$$

hence $\sigma(X,Y) = 0$. As in [8, Proposition 1], one can prove that $\lambda_{rs} = 0$ and the equalities for $C_{rs}$, $D_{rs}$, $B_{rs}$. Writing down the Bianchi identity for the vectors $I_r p$, $I_s q$, $I_t q$ and taking the projection on $\mathbb{H}^n$, we have

$$I_t X_{rs} = I_s X_{rt}.$$

Hence, $X_{rs} = I_s X_{r0}$. Substituting this back to the above equation, we get $I_t I_s X_{r0} = I_s I_t X_{r0}$. Taking $t = 1$, $s = 2$, we obtain $X_{r0} = 0$. This shows that $X_{rs} = 0$. From this and (5) it follows that $\mu_s = 0$.

Writing down the Bianchi identity for the vectors $I_r q$, $I_s q$, $I_t q$ and taking the projection on $\mathbb{H}^n$, we get

$$I_t S_{rs} + I_r S_{st} + I_s S_{tr} = 0.$$

Taking $t = 0$, we have

$$S_{rs} = I_r S_{0s} - I_s S_{0r}.$$

Substituting this to the initial equality and taking $t = 1$, $r = 2$, $s = 3$, we obtain

$$S_{03} = jS_{01} - iS_{02}.$$

Note that $R(X, Y)Z = R'(X, Y)Z - g(Z, Q(X, Y))p$. The Bianchi identity written for the vectors $X$, $Y$, $Z$ implies $R' \in \mathcal{R}(\mathfrak{sp}(n))$. Moreover,

$$g\bigl(Z, Q(X, Y)\bigr) + g\bigl(X, Q(Y, Z)\bigr) + g\bigl(Y, Q(Z, X)\bigr) = 0. \tag{22}$$

Hence,

$$\eta\bigl(Q(X, Y), Z\bigr) + \eta\bigl(Q(Y, Z), X\bigr) + \eta\bigl(Q(Z, X), Y\bigr) = 0. \tag{23}$$

From the Bianchi identity written for the vectors $q$, $X$, $Y$ it follows that

$$P_0(X)Y + Q(X, Y) - P_0(Y)X = 0.$$

This and (23) imply $P_0 \in \mathcal{P}(\mathfrak{h})$. Using (6), we get $R(I_s q, X) = -R(q, I_s X)$, hence

$$P_s(X) = -P_0(I_s X), \qquad T_s(X) = -T_0(I_s X), \qquad \theta_s(X) = -\theta_0(I_s X), \quad s \neq 0.$$

The Bianchi identity written for the vectors $I_r q$, $I_s q$, $X$ implies

$$A_{rs}X + I_r T_s(X) - I_s T_r(X) = 0. \tag{24}$$

Taking $r = 0$, we get $T_s(X) = I_s T_0(X) - A_{0s}(X)$. Substituting this to (24) and taking $r = 1$, $s = 2$, we obtain

$$T_0 = -\frac{1}{2}(I_1 A_{01} + I_2 A_{02} + I_3 A_{03}).$$

Writing down the Bianchi identity for the vectors $I_r q$, $I_s q$, $X$ and taking the projection on $\mathbb{H}p$, we get

$$-g(X, S_{rs}) + I_r\theta_s(X) - I_r\theta_r(X) = 0.$$

Using this it is easy to get (19) and (20). The Bianchi identity applied to $X$, $Y$, $q$ implies the equality for $\tau(X, Y)$ from (17). We have proved that any $R \in \mathcal{R}(\mathfrak{sp}(1, n+1)_{\mathbb{H}p})$ satisfies the conditions of the proposition.

Conversely, it can be checked that any element $R$ satisfying the conditions of the proposition belongs to $\mathcal{R}(\mathfrak{sp}(1, n+1)_{\mathbb{H}p})$. □

Denote by $\mathfrak{sp}(1,1)_{\mathbb{H}p}$ the subalgebra of $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ that annihilates $\mathbb{H}^n \subset \mathbb{H}^{1,n+1}$. The space $\mathcal{R}(\mathfrak{sp}(1,1)_{\mathbb{H}p})$ is found in [8, Proposition 1]. Note that any $R$ given by elements $C_{rs}$, $B_{rs}$, $D_{rs}$ and such that all the rest elements are zero belongs to $\mathcal{R}(\mathfrak{sp}(1,1)_{\mathbb{H}p})$. In particular, we get

**Lemma 1.** *Any subalgebra $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ such that $\dim_{\mathbb{R}} \mathrm{pr}_{\mathbb{H}}\, \mathfrak{g} = 1$ is not a Berger algebra.* □

### 4.2. *The algebras listed in the statement of Theorem 2 are Berger algebras*

Here we prove that the algebras listed in the statement of Theorem 2 are Berger algebras.

Let $\mathfrak{g} = \mathfrak{g}_{I1}$. If $m = n$ and $\mathfrak{h}_0 = \mathfrak{sp}(1)$, then any $R \in \mathcal{R}(\mathfrak{g})$ is given as in Proposition 4.1 with $A_{01}, A_{02}, A_{03} \in \mathfrak{h}$. Since the elements $C_{01} \in \mathbb{H}$, $A_{01} \in \mathfrak{h}$, $S_{01} \in \mathbb{H}^n$, $D_{01} \in \operatorname{Im}\mathbb{H}$ can be chosen in arbitrary way, $\mathfrak{g}$ is a Berger algebra. If $m = n$ and $\mathfrak{h}_0 = \mathbb{R}i$, then from [8, Section 4] it follows that in addition to the above case $C_{01} = 0$ and $C_{02} \in \mathbb{R} \oplus \mathbb{R}i$ is arbitrary, hence $\mathfrak{g}$ is a Berger algebra. Suppose that $m < n$. Then $\mathfrak{h}_0 = \mathbb{R}i$. Each $S_{rs}$ can be written as $S_{rs} = S'_{rs} + S''_{rs}$, where $S'_{rs} \in \mathbb{H}^m$ and $S''_{rs} \in \mathbb{C}^{n-m}$. Then $S''_{01}, S''_{02}, S''_{03} \in \mathbb{C}^{n-m}$. The condition $S''_{03} = jS''_{01} - iS''_{02}$ implies $S''_{01} = 0$, on the other hand, $S''_{02} \in \mathbb{C}^{n-m}$ is arbitrary. This shows that $\mathfrak{g}$ is a Berger algebra.

Let $\mathfrak{g} = \mathfrak{g}_{I2}$. Suppose that $\operatorname{Im}\phi = \mathbb{R}i$. Let $R \in \mathcal{R}(\mathfrak{g})$. From the above example we get that $C_{01} = 0$. In addition, $C_{02} = c_1 + \phi(A_{02})$, where $c_1 \in \mathbb{R}$. Hence, $C_{03} = \phi(A_{02})i + c_1 i$. This shows that $c_1 = -i\phi(A_{03})$. Consequently $\mathfrak{g}$ is a Berger algebra. The case $\operatorname{Im}\phi = \mathfrak{sp}(1)$ will follow from this and the next case.

Let $\mathfrak{g} = \mathfrak{g}_{I4}$ and $R \in \mathcal{R}(\mathfrak{g})$. Let $\phi = i\phi_1 + j\phi_2 + k\phi_3$, where the maps $\phi_1$, $\phi_2$, $\phi_3$ take values in $\mathbb{R}$. Then

$$C_{rs} = \varphi(A_{rs}) + \phi_1(A_{rs})i + \phi_2(A_{rs})j + \phi_3(A_{rs})k.$$

The condition $C_{03} = C_{02}i - C_{01}j$ is equivalent to the equalities

$$\phi_2(A_{01}) - \phi_1(A_{02}) = \varphi(A_{03}), \qquad \varphi(A_{02}) + \phi_3(A_{01}) = \phi_1(A_{03}),$$
$$\phi_3(A_{02}) - \varphi(A_{01}) = \phi_2(A_{03}), \qquad -\phi_2(A_{02}) - \phi_1(A_{01}) = \phi_3(A_{03}).$$

It is not hard to see that these conditions can be satisfied taking appropriate $A_{01}$, $A_{02}$, $A_{03}$. For example, if $\operatorname{Im}\phi = \mathbb{R}i$, then there exists a decomposition $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$ such that $\ker\varphi = \mathfrak{h}_2 \oplus \mathfrak{h}_3$ and $\ker\phi = \mathfrak{h}_1 \oplus \mathfrak{h}_3$. In this case, it is enough to take $A_{01} \in \mathfrak{h}_3$, $A_{02} \in \mathfrak{h}_2$, $A_{03} \in \mathfrak{h}_1$ such that $\varphi(A_{03}) = -\phi_1(A_{02}) = 1$. This shows that $\mathfrak{g}$ is a Berger algebra.

The other Lie algebras from the statement of the theorem can be considered in the same way. For $\mathfrak{g}_{III2}$ and $\mathfrak{g}_{IV}$ note the following. Obviously, $L'$ satisfies condition (28) given below. Let $X, Y, jX - iY \in L'$ and $S_{01} = X$, $S_{02} = Y$, then $S_{03} = jS_{01} - iS_{02} = jX - iY \in L'$, hence $L'$ is spanned by $S_{rs}$.

### 4.3. Weakly-irreducible subalgebras of $\mathfrak{sp}(1, n+1)$ and real vector subspaces in $\mathbb{H}^n$

Now we review the classification of weakly irreducible subalgebras $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ obtained in [7] and make several corrections.

In [7] was constructed a homomorphism $f : \mathfrak{sp}(1, n+1)_{\mathbb{H}p} \to \mathfrak{sim}\mathbb{H}^n$, where $\mathfrak{sim}\mathbb{H}^n = \mathbb{R} \oplus (\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)) \ltimes \mathbb{H}^n$ is the Lie algebra of the group $\mathrm{Sim}\,\mathbb{H}^n$ of similarity transformations of $\mathbb{H}^n$. The homomorphism $f$ is surjective with the kernel $\operatorname{Im}\mathbb{H}$ and it is given as $f(a_0 + a_1, A, X, b) = (a_0, -a_1 + A, X)$, where $a_0 \in \mathbb{R}$, $a_1 \in \mathfrak{sp}(1)$. Let $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ be a weakly irreducible subalgebra and $L = \mathrm{pr}_{\mathbb{H}^n}\,\mathfrak{g} \subset \mathbb{H}^n$. It is shown that $\operatorname{span}_{\mathbb{H}} L = \mathbb{H}^n$ and the connected subgroup of $\mathrm{Sim}\,\mathbb{H}^n$ corresponding to $f(\mathfrak{g}) \subset \mathfrak{sim}\mathbb{H}^n$ preserves $L$ and acts on it transitively. It was stated that there exists a $g$-orthogonal decomposition $L = L_1 \oplus L_2 \oplus L_3$ such that $L_1 \subset \mathbb{H}^n$ is a quaternionic subspace, $L_2 \subset \mathbb{H}^n$ is a real subspace such that $iL_2 = L_2$, $L_2 \cap jL_2 = 0$, $\operatorname{span}_{\mathbb{H}} L_2 = L_2 \oplus jL_2$, and $L_3 \subset \mathbb{H}^n$ is a real subspace such that $L \cap I_s L = 0$, $1 \leqslant s \leqslant 3$. Let $n = 1$ and $e \in \mathbb{H}$ be a non-zero vector, then the vector subspace $L = i\mathbb{R}e \oplus j\mathbb{R}e \oplus k\mathbb{R}e$ is missing. Next, it was stated that $\dim_{\mathbb{R}} L_3 = \dim_{\mathbb{H}} \operatorname{span}_{\mathbb{H}} L_3$. Let $n = 2$, $e_1$, $e_2$ a basis of $\mathbb{H}^2$, then the space $L = \operatorname{span}_{\mathbb{R}}\{e_1, e_2, je_1 + ie_2\}$ is missing. Finally it was stated that there exists a $g$-orthonormal basis of $\mathbb{H}^n$ consisting of vectors from $L_1$, $L_2$ and $L_3$, which is also not true in general.

Let $L \subset \mathbb{H}^n$ be a real subspace such that $\operatorname{span}_{\mathbb{H}} L = \mathbb{H}^n$. Put $L_1 = L \cap iL \cap jL \cap kL$, i.e. $L_1$ is the biggest quaternionic vector subspace in $L$. Let $L_2 = \{X \in L \mid g(X, L_1) = 0\}$, then $L = L_1 \oplus L_2$ and $L_2 \cap iL_2 \cap jL_2 \cap kL_2 = 0$. Note that the subspaces $L_2 \cap iL_2 \cap jL_2$, $L_2 \cap iL_2 \cap kL_2$, $L_2 \cap jL_2 \cap kL_2$, $iL_2 \cap jL_2 \cap kL_2$ can be taken to each other by $i$, $j$, $k$, and the intersection of any two of these subspaces are zero. Let $L_3$ be the direct sum of these subspaces. Let $L_4 = \{X \in L_2 \mid g(X, L_3) = 0\}$ and $L_5 = \{Y \in L_2 \mid g(Y, L_4) = 0\}$. Then $L_2 = L_5 \oplus L_4$ and

$$L_5 = L_3 \cap L_2 = iU \oplus jU \oplus kU = \mathfrak{sp}(1) \cdot U, \quad \mathfrak{sp}(1) = \operatorname{span}_{\mathbb{R}}\{i, j, k\},\ U = iL_2 \cap jL_2 \cap kL_2.$$

By the construction, it holds $I_r L_4 \cap I_s L_4 \cap I_t L_4 = 0$, if $r$, $s$, $t$ are pairwise different. We see that $L_5$ is the biggest subspace of $L_2$ of the form $\mathfrak{sp}(1) \cdot V$, where $V \subset \operatorname{span}_{\mathbb{H}} L_2$ is a real subspace. In particular, this shows that the definition of $L_5$ does not depend on the choice of the generators $I_1$, $I_2$, $I_3$ of $\mathfrak{sp}(1) = \operatorname{span}_{\mathbb{R}}\{I_1, I_2, I_3\}$. Let $h_1 \in \mathfrak{sp}(1)$ be an element with $h_1^2 = -\mathrm{id}$ (any non-zero element of $\mathfrak{sp}(1)$ is proportional to such element). Let $L_4^1 = h_1 L_4 \cap L_4$. Suppose that $L_4^1 \neq 0$, then there is a $g$-orthogonal decomposition $L_4 = L_4^1 \oplus L_4'$ and it holds $h_1 L_4' \cap L_4'$. Taking other $h_2 \in \mathfrak{sp}(1)$, we may decompose $L_4'$. Clearly, this process is finite and we will get a $g$-orthogonal decomposition

$$L_4 = L_4^1 \oplus \cdots \oplus L_4^l \oplus L'$$

such that each $L_4^\alpha$ is $h_\alpha$-invariant for some $h_\alpha \in \mathfrak{sp}(1)$ with $h_\alpha^2 = -\mathrm{id}$ and $hL_4^\alpha \cap L_4^\alpha = 0$ if $h \in \mathfrak{sp}(1)$ is not proportional to $h_\alpha$. Next, $hL' \cap L' = 0$ for any non-zero $h \in \mathfrak{sp}(1)$. Now we change the quaternionic structure $\mathfrak{sp}(1) = \operatorname{span}_{\mathbb{R}}\{I_1, I_2, I_3\}$ on $\mathbb{R}^{4n}$ to another quaternionic structure $\widetilde{\mathfrak{sp}(1)} = \operatorname{span}_{\mathbb{R}}\{\tilde{I}_1, \tilde{I}_2, \tilde{I}_3\}$ such that $\tilde{I}_1|_{\operatorname{span}_{\mathbb{H}} L_4^\alpha} = h_\alpha$. This means that we consider subalgebras of $\mathfrak{sp}(1, n+1)$ up to a conjugancy by elements of $\mathrm{SO}(4, 4n+4)$. After such change we get

$$L = L_1 \oplus L_5 \oplus L_4^1 \oplus L',$$

where $L_1$, $L_5$, $L'$ satisfy the same properties as above and $L_4^1$ is $I_1$-invariant. Let $e_1, \dots, e_m$ be a $g$-orthonormal basis of $L_1 \simeq \mathbb{H}^m$. Let $\mathbb{R}^{m_1} = iL_2 \cap jL_2 \cap kL_2$ and let $\{e_{m+1}, \dots, e_{m+m_1}\}$ be an $\eta$-orthonormal basis of $\mathbb{R}^{m_1}$. Obviously, $L_5 = i\mathbb{R}^{m_1} \oplus j\mathbb{R}^{m_1} \oplus k\mathbb{R}^{m_1}$. Let $X, Y \in \operatorname{span}_{\mathbb{H}} L_4'$. Note that the equality

$$h(X, Y) = \eta(X, Y) + i\eta(X, I_1 Y)$$

defines a Hermitian metric on the complex space $\operatorname{span}_{\mathbb{H}} L_4'$. It holds

$$g(X, Y) = h(X, Y) + h(X, I_2 Y)j.$$

Let $e_{m+m_1+1}, \dots, e_{m+m_1+m_2}$ be an $h$-orthonormal basis of the complex space $L_4'$. Let $w(X, Y) = h(X, I_2 Y)$, then the restriction of $w$ to $L_4'$ is a $\mathbb{C}$-linear skew-symmetric bilinear form.

To describe the structure of $L'$ we use results from [11], where all real subspaces $V$ of quaternionic vector spaces $U$ are found. First a pair $(V, U)$ of such spaces is called indecomposable if there are no pairs $(V_1, U_1)$, $(V_2, U_2)$ such that $U = U_1 \oplus U_2$ and $V = V_1 \oplus V_2$. In our case, $L'$ may be decomposed into a $g$-orthogonal direct sum of real spaces $V$ such that the pair $(V, \operatorname{span}_{\mathbb{H}} V)$ is indecomposable. By our construction, it is enough to consider pairs $(V, \operatorname{span}_{\mathbb{H}} V)$ such that $hV \cap V = 0$ for any $h \in \mathfrak{sp}(1)$. Then we get only the following two possibilities:

$$V = A(2l-1)$$
$$= \operatorname{span}_{\mathbb{R}}\{f_1, \dots, f_{l-1}, f_{l+1}, \dots, f_{2l-1},\ if_1 + jf_2, \dots, if_{l-1} + jf_l,\ f_l + if_{l+1}, jf_{l+1} + if_{l+2}, \dots, jf_{2l-2} + if_{2l-1}\},$$

where $f_1, \dots, f_{2l-1}$ ($l \geqslant 2$) is a basis of $\mathbb{H}^{2l-1}$, and

$$V = B(l) = \operatorname{span}_{\mathbb{R}}\{f_1, \dots, f_l,\ if_1 + jf_2, \dots, if_{l-1} + jf_l\},$$

where $f_1, \dots, f_l$ ($l \geqslant 1$) is a basis of $\mathbb{H}^l$.

We get that $L$ is given by

$$L = \mathbb{H}^m \oplus \operatorname{Im} \mathbb{H}^{m_1} \oplus \mathbb{C}^{m_2} \oplus L', \tag{25}$$

i.e. as in Section 3, but at the moment $L'$ is a $g$-orthogonal direct sum of vector spaces of the form $A(2l-1)$ and $B(l)$.

Summing the above arguments and the results from [8], we obtain the following theorem.

**Theorem 3.** *Let $n \geqslant 1$. Any weakly irreducible subalgebra of $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ that contains the ideal $\operatorname{Im}\mathbb{H}$ is conjugated by an element of $\mathrm{SO}(4, 4n+4)$ to one of the following subalgebras:*

**Type** $\mathbb{R}$. $\mathfrak{g} = \mathbb{R} \oplus \bar{\mathfrak{h}} \ltimes (L \ltimes \operatorname{Im}\mathbb{H})$,
**Type** $\varphi$. $\mathfrak{g} = \{\varphi(A) + A \mid A \in \bar{\mathfrak{h}}\} \ltimes (L \ltimes \operatorname{Im}\mathbb{H})$,
**Type** $\psi$. $\mathfrak{g} = \{A + \psi(A) \mid A \in \bar{\mathfrak{h}}\} \ltimes (W \ltimes \operatorname{Im}\mathbb{H})$,

*where $L$ is as in (25), $\bar{\mathfrak{h}} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ is a subalgebra such that $\tilde{\mathfrak{h}} = \{(-a, A) \mid (a, A) \in \bar{\mathfrak{h}}\} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ preserves $L$; $\varphi : \bar{\mathfrak{h}} \to \mathbb{R}$ is a linear map with $\varphi|_{\bar{\mathfrak{h}}'} = 0$; for the last algebra $L = W \oplus U$ is an orthogonal decomposition, the Lie algebra $\tilde{\mathfrak{h}}$ annihilates $U$, and $\psi : \bar{\mathfrak{h}} \to U$ is a surjective linear map with $\psi|_{\bar{\mathfrak{h}}'} = 0$.*

Note that $\bar{\mathfrak{h}} = \operatorname{pr}_{\mathfrak{sp}(1)\oplus\mathfrak{sp}(n)} \mathfrak{g}$ and $\tilde{\mathfrak{h}} = \operatorname{pr}_{\mathfrak{sp}(1)\oplus\mathfrak{sp}(n)} f(\mathfrak{g})$.

### *4.4. Classification of the Berger algebras containing* $\operatorname{Im}\mathbb{H}$

Let $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ be a weakly irreducible subalgebra, $\bar{\mathfrak{h}} = \operatorname{pr}_{\mathfrak{sp}(1)\oplus\mathfrak{sp}(n)} \mathfrak{g}$, $\tilde{h} = \{(-a, A) \mid (a, A) \in \bar{\mathfrak{h}}\}$ and $L = \operatorname{pr}_{\mathbb{H}^n} \mathfrak{g}$. Then $L \subset \mathbb{H}^n$ is a subspace of the form $L = \mathbb{H}^m \oplus \operatorname{Im}\mathbb{H}^{m_1} \oplus \mathbb{C}^{m_2} \oplus L'$ (see Section 4.3), and $\tilde{\mathfrak{h}}$ preserves $L$. In particular, $\tilde{\mathfrak{h}}$ is contained in the intersection

$$\mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \cap \mathfrak{so}(L) \oplus \mathfrak{so}\big(L^{\perp_\eta}\big). \tag{26}$$

**Lemma 2.** *Let $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ be a weakly irreducible Berger subalgebra. Then the following holds:*

1) *If $L' \neq 0$, then $\bar{\mathfrak{h}} \subset \mathfrak{sp}(m)$, $\operatorname{pr}_{\mathbb{H}} \mathfrak{g} = 0$ and $L'$ is a $g$-orthogonal sum of the spaces of type $B(l)$ with $l \geqslant 2$.*
2) *Suppose that $L' = 0$, then*
   2.a) *if $m_1 \neq 0$ and $m_2 = 0$, i.e. $L = \mathbb{H}^m \oplus \operatorname{Im}\mathbb{H}^{m_1}$, $m + m_1 = n$, then $\operatorname{pr}_{\mathbb{R}} \mathfrak{g} = 0$, $\bar{h} \subset \{a + \operatorname{Op}(aE_{m_1}) \mid a \in \mathfrak{sp}(1)\} \oplus \mathfrak{sp}(m)$, the projection of $\bar{\mathfrak{h}}$ to $\{a + \operatorname{Op}(aE_{m_1}) \mid a \in \mathfrak{sp}(1)\}$ is either trivial or coincides with $\{a + \operatorname{Op}(aE_{m_1}) \mid a \in \mathfrak{sp}(1)\}$;*
   2.b) *if $m_1 = 0$ and $m_2 \neq 0$, i.e. $L = \mathbb{H}^m \oplus \mathbb{C}^{m_2}$, $m + m_2 = n$, then $\bar{h} \subset \mathbb{R}i \oplus \mathfrak{sp}(m)$;*
   2.c) *if $m_1 \neq 0$ and $m_2 \neq 0$, i.e. $L = \mathbb{H}^m \oplus \operatorname{Im}\mathbb{H}^{m_1} \oplus \mathbb{C}^{m_2}$, $m + m_1 + m_2 = n$, then $\bar{h} \subset \mathfrak{sp}(m)$ and $\operatorname{pr}_{\mathbb{H}} \mathfrak{g} = 0$.*

**Proof.** 1) Suppose that $L' \neq 0$. Let $R \in \mathcal{R}(\mathfrak{g})$ be a tensor given as in Proposition 4.1. Then,

$$\operatorname{pr}_{\mathbb{H}\oplus\mathfrak{sp}(n)} R(q, I_s q) = C_{0s} + A_{0s} \in \mathbb{R} \oplus \bar{\mathfrak{h}}.$$

This shows that $\bar{C}_{0s} + A_{0s}$ preserves $L$. It holds $A_{0s} = I_s T_0 + T_0 I_s$. Since $T_0$ takes values in $L$, for any $X \in L$ it holds

$$\operatorname{pr}_{\operatorname{span}_{\mathbb{H}} L'}(\bar{C}_{0s} X + I_s T_0 X) \in L'. \tag{27}$$

Suppose that $L'$ is of type $B(l)$, $l \geqslant 2$ and it is given by vectors $f_1, \dots, f_l$. Then

$$\operatorname{pr}_{\operatorname{span}_{\mathbb{H}} L'} T_0 f_1 = \operatorname{pr}_{L'} T_0 f_1 = a_1 f_1 + \dots + a_l f_l + b_1(if_1 + jf_2) + \dots + b_{l-1}(if_{l-1} + jf_l),$$

where $a_1, \dots, a_l, b_1, \dots, b_{l-1} \in \mathbb{R}$. From (27) it follows that $\bar{C}_{0s} f_1 + I_s \operatorname{pr}_{L'} T_0 f_1 \in L'$. Taking $s = 1$, we get

$$\bar{C}_{01} f_1 + a_1 if_1 + \dots + a_l if_l - b_1 f_1 \dots - b_{l-1} f_{l-1} + kb_1 f_2 + \dots + kb_{l-1} f_l \in L'.$$

Hence, $b_1 = \dots = b_{l-1} = a_2 = \dots = a_l = 0$ and $\bar{C}_{01} = c_1 - a_1 i$ for some $c_1 \in \mathbb{R}$. In particular, $\operatorname{pr}_{L'} T_0 f_1 = a_1 f_1$. Similarly, we get $\bar{C}_{02} = c_2 - a_1 j$ and $\bar{C}_{03} = c_3 - a_1 k$ for some $c_2, c_3 \in \mathbb{R}$. Using the equality $C_{03} = C_{02} i - C_{01} j$, we obtain $a_1 = c_1 = c_2 =$

$c_3=0$. Hence, $C_{rs}=0$. This shows that $\bar{\mathfrak{h}}\subset\mathfrak{sp}(n)$ and $\mathrm{pr}_{\mathbb{H}}\,\mathfrak{g}=0$. For $L'$ of type $A(2l-1)$, $l\geqslant 2$, the proof is similar, hence $\bar{\mathfrak{h}}\subset\mathfrak{sp}(n)$ for any $L'$. Let $\mathfrak{h}=\bar{\mathfrak{h}}$. We claim that $\mathfrak{h}$ preserves decomposition (25). Since $\mathfrak{h}$ commutes with $I_1$, $I_2$, $I_3$, it preserves $\mathbb{H}^m=L\cap I_1L\cap I_2L\cap I_3L$. Hence $\mathfrak{h}$ preserves $(\mathbb{H}^m)^{\perp_\eta}=L_2=\mathrm{Im}\,\mathbb{H}^{m_1}\oplus\mathbb{C}^{m_2}\oplus L'$. Next, $\mathfrak{h}$ preserves $L_2\cap jL_2=i\mathbb{R}^{m_1}\oplus k\mathbb{R}^{m_1}$ and $L_2\cap kL_2=i\mathbb{R}^{m_1}\oplus j\mathbb{R}^{m_1}$, i.e. it preserves $i\mathbb{R}^{m_1}$. Thus $\mathfrak{h}$ preserves $\mathbb{R}^{m_1}$ and $\mathrm{Im}\,\mathbb{H}^{m_1}$. By similar arguments, $\mathfrak{h}$ preserves $\mathbb{C}^{m_2}$ and $L'$. The claim is proved.

The space $\mathbb{H}^n=\mathrm{span}_{\mathbb{H}}\,L$ is the direct sum of four subspaces and $\mathfrak{h}$ preserves this decomposition, hence we may write $A_{rs}=A^1_{rs}+A^2_{rs}+A^3_{rs}+A^4_{rs}$. Similarly, we decompose the elements $S_{rs}$ and $T_s$. Equality (14) shows that $T^1_s=T_s|_{\mathbb{H}^m}$, $T^2_s=T_s|_{\mathbb{H}^{m_1}}$, $T^3_s=T_s|_{\mathbb{H}^{m_2}}$ and $T^4_s=T_s|_{\mathrm{span}_{\mathbb{H}}L'}$. Clearly, these maps take values in $\mathbb{H}^m$, $\mathrm{Im}\,\mathbb{H}^{m_1}$, $\mathbb{C}^{m_2}$ and $L'$, respectively. Since $\mathfrak{h}$ preserves each summand in the direct sums $\mathbb{C}^{m_1}\oplus j\mathbb{C}^{m_1}$, $\mathbb{C}^{m_2}\oplus j\mathbb{C}^{m_2}$ and $L'\oplus iL'$, and acts in each summand simultaneously, according to [14], $R'\in\mathcal{R}(\mathfrak{h}\cap\mathfrak{sp}(m))$ and $P_0\in\mathcal{R}(\mathfrak{h}\cap\mathfrak{sp}(m))$.

Let $S^2_{01}=is_1+js_2+ks_3$, $S^2_{02}=is_6+js_4+ks_5$, where $s_1,\dots,s_6\in\mathbb{R}^{m_1}$. The condition $S^2_{03}=jS^2_{01}-iS^2_{02}\in\mathrm{Im}\,\mathbb{H}^{m_1}$ is equivalent to the equality $s_6=s_2$. The vectors $s_1,\dots,s_5\in\mathbb{R}^{m_1}$ are arbitrary.

Since $S^3_{01},S^3_{02},S^3_{03}\in\mathbb{C}^{m_2}$, and $S^3_{03}=jS^3_{01}-iS^3_{02}$, we see that $S^3_{01}=0$ and $S^3_{02}\in\mathbb{C}^{m_2}$ may be arbitrary.

Since $A_{rs}$ preserves $\mathrm{Im}\,\mathbb{H}^{m_1}$ and $A_{rs}\in\mathfrak{sp}(n)$, it preserves $i\,\mathrm{Im}\,\mathbb{H}^{m_1}\cap j\,\mathrm{Im}\,\mathbb{H}^{m_1}\cap k\,\mathrm{Im}\,\mathbb{H}^{m_1}=\mathbb{R}^{m_1}$. Let $X\in\mathbb{R}^{m_1}$, then

$$T^2_1(X)=-\frac{1}{2}\big(A^2_{01}(X)+I_3A^2_{02}(X)-I_2A^2_{03}(X)\big)\in\mathrm{Im}\,\mathbb{H}^{m_1},$$

hence, $A^2_{01}(X)=0$. Since $A^2_{01}\in\mathfrak{sp}(m_1)$, this implies $A^2_{01}=0$. Similarly, $A^2_{02}=A^2_{03}=0$.

Let $X\in\mathbb{C}^{m_2}$. Since

$$T^3_0(X)=-\frac{1}{2}\big(I_1A^3_{01}(X)+I_2A^3_{02}(X)+I_3A^3_{03}(X)\big)\in\mathbb{C}^{m_2},$$

$I_2A^3_{02}(X)+I_3A^3_{03}(X)=I_2(A^3_{02}(X)-I_1A^3_{03}(X))$ and $A^3_{0r}(X)\in\mathbb{C}^{m_2}$ for any $r$, we get that $A^3_{02}(X)=I_1A^3_{03}(X)$. This implies $A^3_{02}|_{\mathbb{C}^{m_2}}=I_1A^3_{03}|_{\mathbb{C}^{m_2}}$. Hence, $A^3_{02}|_{\mathbb{C}^{m_2}}=A^3_{03}|_{\mathbb{C}^{m_2}}=0$ and $A^3_{02}=A^3_{03}=0$. Next, $T^3_2(X)=I_2T^3_0(X)-A^3_{02}(X)=\frac{1}{2}I_3A^3_{01}(X)$. Hence, $A^3_{01}(X)=0$. This shows that $A^3_{rs}=0$.

Let $Y\in L'$. Then for $s\neq 0$, $T^4_s(Y)=I_sT^4_0(Y)-A^4_{0s}(Y)\in L'$. Since $L'\cap I_sL'=0$ and $A^4_{0s}$ preserves $L'$, we get $T^4_0(Y)=0$. From (5) applied to the vectors $I_sq$, $X,Y\in\mathbb{H}^n$, $q$, it follows that

$$\eta\big(T_s(X),Y\big)=\eta\big(I_sT_0(Y),X\big).$$

Let $Y\in L'$, we get $\eta(T^4_s(X),Y)=0$ for any $X\in\mathbb{H}^n$ and any $s$. Hence $T^4_s=0$. Consequently, $A^4_{rs}=0$. Thus, $\mathfrak{h}\subset\mathfrak{sp}(m)$.

We see that $L'$ must be spanned by elements $S_{01},S_{02},S_{03}\in L'$ that satisfy $S_{03}=jS_{01}-iS_{02}$, i.e. $L'$ must satisfy

$$L'=\rho\big(L'\big),\quad \text{where } \rho\big(L'\big)=\mathrm{span}_{\mathbb{R}}\big\{X,Y,jX-iY\;\big|\;X,Y,jX-iY\in L'\big\}. \tag{28}$$

Clearly, the space $B(l)$, $l\geqslant 2$ satisfies this condition, while the space $B(1)$ does not satisfy this condition.

**Lemma 3.** *The space $L'=A(2l-1)$, $l\geqslant 2$ does not satisfy the condition (28).*

**Proof.** It can be directly checked that $\rho(A(3))=0$. We claim that if $L'=A(2l-1)$, $l\geqslant 3$, then

$$\rho\big(L'\big)=\mathrm{span}_{\mathbb{R}}\{f_1,\dots,f_{l-1},f_{l+1},\dots,f_{2l-1},if_1+jf_2,\dots,if_{l-2}+jf_{l-1},jf_{l+1}+if_{l+2},\dots,jf_{2l-2}+if_{2l-1}\}.$$

We prove this claim using the induction over $l$. For $l=3$ this can be checked directly. Suppose that the claim holds for some $l\geqslant 3$. We will prove it for $l+1$.

Clearly, $A(2(l+1)-1)$ can be obtained from $A(2l-1)$ adding some vectors $f_0$, $f_{2l}$, $if_0+jf_1$, $jf_{2l-1}+if_{2l}$. Let $X,Y\in A(2(l+1)-1)$. Then,

$$X=af_0+bf_{2l}+c(if_0+jf_1)+d(jf_{2l-1}+if_{2l})+\tilde{X},$$
$$Y=xf_0+yf_{2l}+u(if_0+jf_1)+v(jf_{2l-1}+if_{2l})+\tilde{Y}$$

for some $a,b,c,d,x,y,u,v\in\mathbb{R}$, $\tilde{X},\tilde{Y}\in A(2l-1)$. It can be checked that if $jX-iY\in A(2(l+1)-1)$, then $a,b,c,d=0$ and

$$jX-iY=j\tilde{X}_1-i\tilde{Y}-x(if_0+jf_1)-y(jf_{2l-1}+if_{2l})-uf_2-vf_{2l-2}+uf_0+vf_{2l},$$

where

$$\tilde{X}_1=\tilde{X}+xf_1-yf_{2l-1}-u(if_1+jf_2)-v(jf_{2l-2}+if_{2l-1})\in A(2l-1).$$

Hence, $j\tilde{X}_1-i\tilde{Y}\in A(2l-1)$. This and the induction hypothesis prove the inclusion $\subset$, the inverse inclusion is obvious. The lemma is proved. □

Thus $L'$ is a $g$-orthogonal sum of the spaces of the form $B(l)$, $l \geqslant 2$.

2.a) Suppose that $L' = 0$, $m_1 \neq 0$ and $m_2 = 0$, i.e. $L = \mathbb{H}^m \oplus \operatorname{Im}\mathbb{H}^{m_1}$, $m + m_1 = n$. For simplicity of the exposition we may assume that $m = 0$, i.e. $L = \operatorname{Im}\mathbb{H}^n$. Obviously, $\tilde{\mathfrak{h}} \cap \mathfrak{sp}(1) = 0$ and elements of the form $a - \operatorname{Op}(aE_n)$ (where $a \in \mathfrak{sp}(1)$) preserve $L = \operatorname{Im}\mathbb{H}^n$, hence $\tilde{\mathfrak{h}} \subset \{a - \operatorname{Op}(aE_n) \mid a \in \mathfrak{sp}(1)\} \oplus \mathfrak{h}_1$, where $\mathfrak{h}_1 \subset \mathfrak{sp}(n)$ is a vector subspace preserving $\operatorname{Im}\mathbb{H}^n$. Let $R \in \mathcal{R}(\mathfrak{g})$ be as in Proposition 4.1. Then

$$\operatorname{pr}_{\mathbb{H}\oplus\mathfrak{sp}(n)} R(q, I_s q) = C_{0s} + A_{0s} = C_{0s} + \operatorname{Op}(a_{0s}E_n) + B_{0s},$$

where $a_{0s} = \operatorname{Im} C_{0s}$ and $B_{0s} \in \mathfrak{sp}(n)$ preserves $\operatorname{Im}\mathbb{H}^n$. Clearly, $B_{0s}$ preserves $\mathbb{R}^n$. Recall that

$$T_0(X) = -\frac{1}{2}(I_1 A_{01} X + I_2 A_{02} X + I_3 A_{03} X) \in L$$

for any $X \in \mathbb{H}^n$. Let $e_\alpha \in \mathbb{R}^n \subset \mathbb{H}^n$ be an element of the basis. The condition $T_0(e_\alpha) \in L$ implies $\operatorname{Re}(ia_{01} + ja_{02} + ka_{03}) = 0$. The condition $T_0(ie_\alpha) \in L$ implies

$$B_{01} e_\alpha = \operatorname{Re}\big((ia_{01} + ja_{02} + ka_{03})i\big)e_\alpha.$$

Since $B_{01} \in \mathfrak{sp}(n) \subset \mathfrak{so}(4n)$, we conclude $\operatorname{Re}((ia_{01} + ja_{02} + ka_{03})i) = 0$ and $B_{01} = 0$. Similarly, $B_{02} = B_{03} = 0$ and

$$\operatorname{Re}\big((ia_{01} + ja_{02} + ka_{03})j\big) = \operatorname{Re}\big((ia_{01} + ja_{02} + ka_{03})k\big) = 0.$$

Thus, $ia_{01} + ja_{02} + ka_{03} = 0$, i.e. $a_{03} = ja_{01} - ia_{02}$. This and the equalities $a_{0s} = \operatorname{Im} C_{0s}$, $C_{03} = C_{02}i - C_{01}j$ imply $a_{0s} = C_{0s} \in \operatorname{Im}\mathbb{H}$. Thus since $\mathfrak{g}$ is a Berger algebra, $\operatorname{pr}_{\mathbb{R}}\mathfrak{g} = 0$. Lemma 1 shows that either $\bar{\mathfrak{h}} = 0$, or $\bar{\mathfrak{h}} = \{a + \operatorname{Op}(aE_n) \mid a \in \mathfrak{sp}(1)\}$. If we do not assume that $m = 0$, then $\bar{\mathfrak{h}} \subset \{a + \operatorname{Op}(aE_{m_1}) \mid a \in \mathfrak{sp}(1)\} \oplus \mathfrak{sp}(m)$ and $\operatorname{pr}_{\mathbb{R}}\mathfrak{g} = 0$, moreover, the projection of $\bar{\mathfrak{h}}$ to $\{a + \operatorname{Op}(aE_{m_1}) \mid a \in \mathfrak{sp}(1)\}$ is either trivial or it coincides with $\{a + \operatorname{Op}(aE_{m_1}) \mid a \in \mathfrak{sp}(1)\}$.

2.b) Suppose that $m_1 = 0$ and $m_2 \neq 0$, i.e. $L = \mathbb{H}^m \oplus \mathbb{C}^{m_2}$, $m + m_2 = n$. As above, suppose that $m = 0$, then $L = \mathbb{C}^n$. Let $e_1, \ldots, e_n$ a basis of the complex vector space $L = \mathbb{C}^n$. Let $C + A \in \bar{\mathfrak{h}}$, where $C \in \mathfrak{sp}(1)$ and $A \in \mathfrak{sp}(n)$. Let $A_{\alpha\beta}$ be the matrix of $A$ with respect to the basis $e_1, \ldots, e_n$ of $\mathbb{H}^n$. Then since $(-C + A)e_\alpha \in L$ and $(-C + A)ie_\alpha \in L$, we get $-C + A_{\alpha\alpha} \in \mathbb{C}$ and $-Ci + iA_{\alpha\alpha} \in \mathbb{C}$. Consequently, $C \in \mathbb{C}$. This shows that $A$ preserves $L$ and $\bar{\mathfrak{h}} \subset \mathbb{R}i \oplus \mathfrak{sp}(n)$. Let $R \in \mathcal{R}(\mathfrak{g})$ be as in Proposition 4.1. Then

$$\operatorname{pr}_{\mathbb{H}\oplus\mathfrak{sp}(n)} R(q, I_s q) = C_{0s} + A_{0s} \in \mathbb{C} \oplus \mathfrak{sp}(n)$$

preserves $L$ and $A_{0s}$ preserves $L$. By the arguments of the proof of statement 1), $A_{0s} = 0$. Thus, $\bar{\mathfrak{h}} \subset \mathbb{R}i$. If $m \neq 0$, then $\bar{\mathfrak{h}} \subset \mathbb{R}i \oplus \mathfrak{sp}(m)$.

2.c) Suppose that $m_1 \neq 0$ and $m_2 \neq 0$, i.e. $L = \mathbb{H}^m \oplus \operatorname{Im}\mathbb{H}^{m_1} \oplus \mathbb{C}^{m_2}$, $m + m_1 + m_2 = n$. As in the proof of 2.b), it can be shown that $\bar{\mathfrak{h}} \subset \mathbb{R}i \oplus \mathfrak{sp}(m) \oplus \mathfrak{sp}(m_1)$, i.e. $\operatorname{pr}_{\mathfrak{sp}(1)}\mathfrak{g} \subset i\mathbb{R}$. As in the proof of 2.a), it can be proved that $\operatorname{pr}_{\mathbb{R}}\mathfrak{g} = 0$. From this and Lemma 1 it follows that $\operatorname{pr}_{\mathbb{H}}\mathfrak{g} = 0$, i.e. $\bar{\mathfrak{h}} \subset \mathfrak{sp}(m) \oplus \mathfrak{sp}(m_1)$. By the arguments of the proof of 1), $\bar{\mathfrak{h}} \subset \mathfrak{sp}(m)$. The lemma is proved. □

Now using Theorem 3 and Lemmas 1, 2, it is easy to obtain the classification of weakly irreducible Berger subalgebras of $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ containing $\operatorname{Im}\mathbb{H}$. All such subalgebras are exhausted by the Lie algebras given in the statement of the Theorem 2. Let us consider some examples.

Let $L = \mathbb{H}^m \oplus \operatorname{Im}\mathbb{H}^{m_1} \oplus \mathbb{C}^{m_2} \oplus L'$ and $L' \neq 0$. In this case, for any weakly irreducible Berger subalgebra $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ by Lemma 2, we have $\bar{\mathfrak{h}} \subset \mathfrak{sp}(m)$ and $\operatorname{pr}_{\mathbb{H}}\mathfrak{g} = 0$. Then $\mathfrak{g}$ can be of Type $\varphi$ or Type $\psi$ from Theorem 3. If $\mathfrak{g}$ is of Type $\varphi$, then we get that $\mathfrak{g} = \mathfrak{g}_{II}$ from Theorem 2 with $\varphi = 0$. If $\mathfrak{g}$ is of Type $\psi$, then $\mathfrak{g} = \mathfrak{g}_{IV}$. Next we may assume that $L' = 0$.

Let $L = \mathbb{H}^m \oplus \operatorname{Im}\mathbb{H}^{m_1}$. By Lemma 2, $\operatorname{pr}_{\mathbb{R}}\mathfrak{g} = 0$ and either $\bar{\mathfrak{h}} = \{a + \operatorname{Op}(aE_{m_1}) \mid a \in \operatorname{Im}\mathbb{H}\} \oplus \mathfrak{h}$, or $\bar{\mathfrak{h}} = \{\phi(A) + \operatorname{Op}(\phi(A)E_{m_1}) + A \mid A \in \mathfrak{h}\}$, where $\mathfrak{h} \subset \mathfrak{sp}(m)$ is a subalgebra and $\varphi : \mathfrak{h} \to \mathfrak{sp}(1)$ is a homomorphism. Using the fact that there are no two-dimensional subalgebras of $\mathfrak{sp}(1)$ and Lemma 1, we get that either $\phi = 0$, or $\operatorname{Im}\phi = \mathfrak{sp}(1)$. Since $\operatorname{pr}_{\mathbb{R}}\mathfrak{g} = 0$, $\mathfrak{g}$ is either of Type $\varphi$ (with $\varphi = 0$) or of Type $\psi$. In the first case $\mathfrak{g}$ coincides with $\mathfrak{g}_{III1}$ or $\mathfrak{g}_{III2}$.

Let $\mathfrak{g}$ be of Type $\psi$. Since $\bar{h}$ annihilates the subspace $U \subset L$, it cannot contain $\mathfrak{sp}(1)$, i.e. $\bar{\mathfrak{h}} = \{\phi(A) + \operatorname{Op}(\phi(A)E_{m_1}) + A \mid A \in \mathfrak{h}\}$. If $\phi = 0$, then $\mathfrak{g} = \mathfrak{g}_{IV}$.

Assume that $\phi \neq 0$, then $\phi$ is surjective. Now we find subspaces $U \subset L$ such that $\tilde{\mathfrak{h}}U = 0$. Any $u \in U$ has the form $u = u_1 + u_2$, where $u_1 \in \mathbb{H}^m$, $u_2 \in \operatorname{Im}\mathbb{H}^{m_1}$. Let $\xi = -\phi(A) + \operatorname{Op}(\phi(A)E_{m_1}) + A \in \tilde{\mathfrak{h}}$, then $\xi u_1 = \xi u_2 = 0$. We see that if $u_2 = \sum_{i=m+1}^n a_i e_i$, where $e_{m+1}, \ldots, e_n$ is a basis of $\mathbb{R}^{m_1}$, $a_i \in \operatorname{Im}\mathbb{H}$, then $\xi u_2 = -\sum_{i=m+1}^n [\phi(A), a_i]e_i = 0$. From here $a_i = 0$ for all $i = m+1, \ldots, n$, i.e. $u_2 = 0$. Thus, $u \in \mathbb{H}^m$ and $U \subset \mathbb{H}^m$.

We claim that for any $\xi \in \mathfrak{sp}(1)$, $\xi \neq 0$ it holds $\xi U \cap U = 0$. Indeed, let $\xi U \cap U \neq 0$, then there exists a non-zero vector $u \in U$ such that $\xi u \in U$. Since $\tilde{\mathfrak{h}}$ annihilates $u$ and $\xi u$,

$$\big(\phi(A) - \operatorname{Op}\big(\phi(A)E_{m_1}\big) - A\big)u = 0, \qquad \big(\phi(A) - \operatorname{Op}\big(\phi(A)E_{m_1}\big) - A\big)\xi u = 0$$

for all $A \in \mathfrak{h}$. This implies that $[\xi, \phi(A)]u = 0$ hence, $[\xi, \phi(A)] = 0$ for any $A \in \mathfrak{h}$. Consequently, $[\xi, \mathfrak{sp}(1)] = 0$, where $\xi \in \mathfrak{sp}(1)$. It gives the contradiction. Thus, according to [11], $U$ is a $g$-orthogonal direct sum of vector spaces of the form

$A(2l-1)$ and $B(l)$, $l \geqslant 1$. Further, assume that $U$ contains a subspace of the form $A(2l-1)$ or $B(l)$, where $l \geqslant 2$. Then $f_1, f_2, if_1 + jf_2 \in U$ and

$$\big(\phi(A) - \mathrm{Op}\big(\phi(A)E_{m_1}\big) - A\big) f_1 = 0, \qquad \big(\phi(A) - \mathrm{Op}\big(\phi(A)E_{m_1}\big) - A\big) f_2 = 0,$$
$$\big(\phi(A) - \mathrm{Op}\big(\phi(A)E_{m_1}\big) - A\big)(if_1 + jf_2) = 0$$

for all $A \in \mathfrak{h}$. Hence, $Af_1 = \phi(A)f_1$ and $Af_2 = \phi(A)f_2$. We get $\phi(A)i = i\phi(A)$ and $\phi(A)j = j\phi(A)$ for any $A \in \mathfrak{h}$. Consequently, $\phi = 0$. This gives a contradiction. Thus $U$ is a $g$-orthogonal sum of vector subspaces of the form $B(1)$. Consider $\mathrm{span}_{\mathbb{H}} U \subset \mathbb{H}^m$ and denote by $\mathbb{H}^{m_0}$ its orthogonal complement. Then $\mathbb{H}^m = \mathbb{H}^{m_0} \oplus \mathrm{span}_{\mathbb{H}} U$. Let $e_1, \ldots, e_{m_0}$ be a $g$-orthonormal basis of $\mathbb{H}^{m_0}$. Since $U$ is a direct sum of subspaces of the form $B(1) = \mathbb{R}f_1$, $U = \mathrm{span}_{\mathbb{R}}\{e_{m_0+1}, \ldots, e_m\} = \mathbb{R}^{m-m_0}$, and the vectors $e_{m_0+1}, \ldots, e_m$ are $g$-orthogonal.

Now let us find the algebras $\tilde{\mathfrak{h}}$ that annihilate $U$. Let $\phi(A) - \mathrm{Op}(\phi(A)E_{m_1}) - A \in \tilde{\mathfrak{h}}$. Then $(\phi(A) - A)e_i = 0$, $i = m_0 + 1, \ldots, m$, i.e. $Ae_i = \phi(A)e_i$. This shows that $A|_{\mathbb{H}^{m-m_0}} = \mathrm{Op}(\phi(A)E_{m-m_0})$. Moreover, $A$ preserves $\mathbb{H}^{m-m_0}$, and, consequently, it preserves $\mathbb{H}^{m_0}$. It is clear that the obtained properties of $\tilde{\mathfrak{h}}$ are equivalent to the conditions

$$\tilde{\mathfrak{h}} \subset \big\{a - \mathrm{Op}(aE_{m-m_0}) - \mathrm{Op}(aE_{m_1}) \bigm| a \in \mathfrak{sp}(1)\big\} \oplus \mathfrak{sp}(m_0), \quad \tilde{h} \not\subset \mathfrak{sp}(m_0),$$

or to the conditions

$$\bar{\mathfrak{h}} \subset \big\{a + \mathrm{Op}(aE_{m-m_0}) + \mathrm{Op}(aE_{m_1}) \bigm| a \in \mathfrak{sp}(1)\big\} \oplus \mathfrak{sp}(m_0), \quad \bar{h} \not\subset \mathfrak{sp}(m_0).$$

Suppose that $\bar{\mathfrak{h}} = \{a + \mathrm{Op}(aE_{m-m_0}) + \mathrm{Op}(aE_{m_1}) \mid a \in \mathfrak{sp}(1)\} \oplus \mathfrak{h}$, where $\mathfrak{h} \subset \mathfrak{sp}(m_0)$. Since $\psi|_{\tilde{h}'} = 0$, $\psi$ is zero on the first summand. We see that $\mathfrak{g} = \mathfrak{g}_{V1}$. If $\bar{\mathfrak{h}}$ does not contain $\{a + \mathrm{Op}(aE_{m-m_0}) + \mathrm{Op}(aE_{m_1}) \mid a \in \mathfrak{sp}(1)\}$, then $\mathfrak{g} = \mathfrak{g}_{V2}$.

Further, let $L = \mathbb{H}^m \oplus \mathbb{C}^{m_2}$. By Lemma 2, $\bar{\mathfrak{h}} \subset \mathbb{R}i \oplus \mathfrak{sp}(m)$. Assume that $\bar{\mathfrak{h}} = \mathbb{R}i \oplus \mathfrak{h}$, $\mathfrak{h} \subset \mathfrak{sp}(m)$. In this case, $\mathfrak{g}$ can be of Type $\mathbb{R}$ or Type $\varphi$. Let $\mathfrak{g}$ be of Type $\mathbb{R}$, then $\mathfrak{g} = \mathfrak{g}_{I1}$. If $\mathfrak{g}$ is of Type $\varphi$ and $\varphi \neq 0$, we have $\{\varphi(A) + A \mid A \in \bar{\mathfrak{h}}\} = \mathbb{R}(\varphi(i) + i) \oplus \{\varphi(A) + A \mid A \in \mathfrak{h}\}$, $\mathfrak{h} \subset \mathfrak{sp}(m)$. If $\varphi(i) = 0$, then $\mathfrak{g} = \mathfrak{g}_{I3}$. If $\varphi(i) \neq 0$, then $\mathfrak{g} = \mathfrak{g}_{I5}$. Suppose that $\bar{\mathfrak{h}} = \{\varphi(A)i + A \mid A \in \mathfrak{h}\}$, $\mathfrak{h} \subset \mathfrak{sp}(m)$. Again, $\mathfrak{g}$ can be of Type $\mathbb{R}$ or Type $\varphi$. In the first case, $\mathfrak{g} = \mathfrak{g}_{I2}$. In the second case

$$\big\{\varphi(A) + A \bigm| A \in \bar{\mathfrak{h}}\big\} = \big\{\varphi\big(\phi(A)i + A\big) + \phi(A)i + A \bigm| A \in \mathfrak{h}\big\} = \big\{\widetilde{\varphi}(A) + \phi(A)i + A \bigm| A \in \mathfrak{h}\big\},$$

here $\widetilde{\varphi}(A) = \varphi(\phi(A)i + A)$, $\mathfrak{h} \subset \mathfrak{sp}(m)$. By Lemma 1, $\widetilde{\varphi}$ and $\phi$ are not proportional. Thus, $\mathfrak{g} = \mathfrak{g}_{I4}$. Now let $\bar{\mathfrak{h}} \subset \mathfrak{sp}(m)$. Then $\mathfrak{g}$ can be of Type $\varphi$. Using Lemma 1, we get that $\mathfrak{g} = \mathfrak{g}_{II}$ with $\varphi = 0$. Also, $\mathfrak{g}$ can be of Type $\psi$, then $\mathfrak{g} = \mathfrak{g}_{IV}$.

All other cases can be considered in the similar way.

*Any weakly irreducible Berger subalgebra* $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ *contains* $\mathrm{Im}\,\mathbb{H}$.

**Proposition 4.2.** *Let* $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ *be a weakly irreducible Berger subalgebra, then it is conjugated to a subalgebra that contains the ideal* $\mathrm{Im}\,\mathbb{H}$.

**Proof.** Let $\mathfrak{g} \subset \mathfrak{sp}(1, n+1)_{\mathbb{H}}p$ be a weakly irreducible subalgebra. Let $f : \mathfrak{sp}(1, n+1)_{\mathbb{H}p} \to \mathfrak{sim}\mathbb{H}^n$ be the homomorphism as in Section 4.3. According to [7] and Section 4.3, the image $f(\mathfrak{g}) \subset \mathfrak{sim}\mathbb{H}^n$ coincides with one of the following algebras:

**Type** $\mathbb{R}$. $f(\mathfrak{g}) = (\mathbb{R} \oplus \tilde{\mathfrak{h}}) \ltimes L$,
**Type** $\varphi$. $f(\mathfrak{g}) = \{\varphi(A) + A \mid A \in \tilde{\mathfrak{h}}\} \ltimes L$,
**Type** $\psi$. $f(\mathfrak{g}) = \{A + \psi(A) \mid A \in \tilde{\mathfrak{h}}\} \ltimes W$,

where $L$ is as in (25), $\tilde{\mathfrak{h}} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ is a subalgebra, $\tilde{\mathfrak{h}}L \subset L$; $\varphi \in \mathrm{Hom}(\tilde{\mathfrak{h}}, \mathbb{R})$, $\varphi|_{\tilde{\mathfrak{h}}'} = 0$; for the last algebra we have an orthogonal decomposition $L = W \oplus U$, $\tilde{\mathfrak{h}}$ annihilates $U$ and $\psi : \tilde{\mathfrak{h}} \to U$ is surjective linear map with $\psi|_{\tilde{\mathfrak{h}}'} = 0$.

Suppose that $\mathfrak{g}$ is a Berger algebra. The structure of the Lie brackets of $\mathfrak{sp}(1, n+1)_{\mathbb{H}p}$ (see Section 3) shows that if $m \neq 0$, or $m_1 \neq 0$, then $\mathfrak{g}$ contains $\mathrm{Im}\,\mathbb{H}$. Thus, $L = \mathbb{C}^{m_2} \oplus L'$. By Lemma 2, if $\mathfrak{g}$ is a Berger algebra with such $L$, then $\mathrm{pr}_{\mathfrak{sp}(n)}\,\mathfrak{g} = 0$. Hence, if $L' = 0$, then $\bar{\mathfrak{h}} \subset \mathbb{R}i$; if $L' \neq 0$, then $\bar{\mathfrak{h}} = 0$. Recall that since $\mathfrak{g}$ is a Berger algebra, $\dim \mathrm{pr}_{\mathbb{H}}\,\mathfrak{g} \neq 1$. This shows that either $f(\mathfrak{g})$ is of Type $\mathbb{R}$ with $\bar{\mathfrak{h}} = \mathbb{R}i$ and $L' = 0$, or $f(\mathfrak{g})$ is of Type $\varphi$ with $\varphi = 0$ and $\bar{\mathfrak{h}} = 0$.

Consider the first case. We have $(1, 0, 0, b) \in \mathfrak{g}$ for some $b \in \mathrm{Im}\,\mathbb{H}$. Let $X \in \mathbb{C}^n$, then $(0, 0, X, c) \in \mathfrak{g}$. Next,

$$\big[(1, 0, 0, b), (0, 0, X, c)\big] = (0, 0, X, 2c) \in \mathfrak{g}.$$

This shows that $\mathbb{C}^n \subset \mathfrak{g}$. Consequently,

$$\big[(0, 0, e_1, 0), (0, 0, ie_1, 0)\big] = (0, 0, 0, -2i) \in \mathfrak{g}.$$

Hence, $\mathbb{R}(0, 0, 0, i) \subset \mathfrak{g}$. If $(0, 0, 0, \alpha j + \beta k) \in \mathfrak{g}$ for some $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 \neq 0$, then taking the Lie bracket of this element with $(i, 0, 0, c) \in \mathfrak{g}$, we get $(0, 0, 0, \alpha k - \beta j) \in \mathfrak{g}$, i.e. $\mathrm{Im}\,\mathbb{H} \subset \mathfrak{g}$. Assume that $\mathfrak{g} \cap \mathbb{R}j \oplus \mathbb{R}k = 0$. Then it is not hard to see that

$$\mathfrak{g} = \mathbb{R}(1, 0, 0, \alpha j + \beta k) \oplus \mathbb{R}(i, 0, 0, -\beta j + \alpha k) \oplus \mathbb{C}^n \ltimes \mathbb{R}(0, 0, 0, i).$$

Let $R \in \mathcal{R}(\mathfrak{g})$. Above we have seen that the elements defining $R$ are zero possibly except for $C_{02} \in \mathbb{C}$, $S_{02} \in \mathbb{C}^n$ and some of $D_{rs}$. Let $X \in \mathbb{H}^n$. It holds $R(I_s q, X) = (0, 0, 0, \theta_s(X)) \in \mathfrak{g}$. Hence, $\theta_s(X) \in \mathbb{R}i$. From (21) it follows that $\eta(\theta_s(X)p, I_2 q) = \eta(I_s S_{02}, X)$. Consequently, $S_{02} = 0$, and $\mathfrak{g}$ is not a Berger algebra.

Suppose that $f(\mathfrak{g})$ is of Type $\varphi$ and $m_2 \neq 0$. Then for some $b, c \in \operatorname{Im}\mathbb{H}$, $(0, 0, e_1, b)$, $(0, 0, ie_1, c) \in \mathfrak{g}$. Taking the Lie brackets of these elements, we get $(0, 0, 0, i) \in \mathfrak{g}$. Using (21), we obtain that $\eta(\theta_s(X)p, I_2 q) = \eta(I_s S_{02}, X)$. If $\mathfrak{g}$ is a Berger algebra, then for some $R \in \mathcal{R}(\mathfrak{g})$ it holds $S_{02} \neq 0$. There exists $X \in \operatorname{Im}\mathbb{H}$ such that $\eta(S_{02}, X) \neq 0$. Hence, $\theta_0(X) \in \operatorname{Im}\mathbb{H}$ has a non-zero projection to $\mathbb{R}j \subset \operatorname{Im}\mathbb{H}$. Since $R(I_s q, X) = (0, 0, 0, \theta_s(X)) \in \mathfrak{g}$, there exists $\alpha, \beta \in \mathbb{R}$ such that $\alpha \neq 0$ and $(0, 0, 0, \alpha j + \beta k) \in \mathfrak{g}$. We may assume that $\alpha^2 + \beta^2 = 1$. In [8, Lemma 6] it is shown that there exists $x, y \in \mathbb{R}$ such that $x^2 + y^2 = 1$ and with respect to the new basis with $p' = (x + iy)p$ and $q' = (x + iy)q$ the elements $(0, 0, 0, i) \in \mathfrak{g}$ and $(0, 0, 0, \alpha j + \beta k) \in \mathfrak{g}$ have the form $(0, 0, 0, i)$ and $(0, 0, 0, j)$, respectively. Note that $S_{03} = -iS_{02} \neq 0$. As above, there exists $X \in \operatorname{Im}\mathbb{H}$ such that $\eta(I_1 S_{03}, X) \neq 0$. Hence, $\eta(\theta_s(X)p, I_3 q) \neq 0$, i.e. $\theta_1(X) \in \operatorname{Im}\mathbb{H}$ has a non-zero projection to $\mathbb{R}k \subset \operatorname{Im}\mathbb{H}$. We conclude that $\mathfrak{g}$ contains $\operatorname{Im}\mathbb{H}$.

Suppose now that $m_2 = 0$, i.e. $L = L'$. Suppose that $\dim \mathfrak{g} \cap \operatorname{Im}\mathbb{H} = 2$. From the [8, Lemmas 5, 6] it follows that choosing an appropriate basis we may get $\mathfrak{g} \cap \operatorname{Im}\mathbb{H} = \mathbb{R}i \oplus \mathbb{R}j$. Since $R(I_s q, X) = (0, 0, 0, \theta_s(X))$, we see that $\theta_s(X) \in \mathbb{R}i \oplus \mathbb{R}j$ for any $X \in \mathbb{H}^n$. Since $\theta_s(X) = g(X, S_{0s}) + I_s\theta_0(X)$, we obtain that for any $X \in \mathbb{H}^n$ and $s = 1, 2$ it holds $g(X, S_{0s}) \in \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j$. Since $g(kS_{0s}, S_{0s}) = kg(S_{0s}, S_{0s}) \in \mathbb{R}k$, we have $g(S_{0s}, S_{0s}) = 0$, consequently, $S_{0s} = 0$ for $s = 1, 2$. This implies $S_{rs} = 0$. Hence, $\mathfrak{g}$ is not a Berger algebra. The case $\dim \mathfrak{g} \cap \operatorname{Im}\mathbb{H} < 2$ follows from this one. This proves the proposition and the theorem. □

## 5. Pseudo-hyper-Kählerian symmetric spaces of index 4

In [1,21,22] indecomposable simply connected pseudo-hyper-Kählerian symmetric spaces of signature $(4, 4n + 4)$ are classified. Here we use the results of this paper to give new proof to this result. For $n = 0$ such new proof is obtained in [8].

As it is explained e.g. in [2] the classification of indecomposable simply connected pseudo-hyper-Kählerian symmetric spaces is equivalent to the classification of pairs $(\mathfrak{g}, R)$ (*symmetric pairs*), where $\mathfrak{g} \subset \mathfrak{sp}(1, n + 1)$ is a subalgebra, $R \in \mathcal{R}(\mathfrak{g})$, $R(\mathbb{R}^{4,4n+4}, \mathbb{R}^{4,4n+4}) = \mathfrak{g}$, and for any $\xi \in \mathfrak{g}$ it holds

$$\xi \cdot R = 0, \qquad (\xi \cdot R)(x, y) = \big[\xi, R(x, y)\big] - R(\xi x, y) - R(x, \xi y), \tag{29}$$

where $x, y \in \mathbb{R}^{4,4n+4}$. *An isomorphism* of symmetric pairs $f : (\mathfrak{g}_1, R_1) \to (\mathfrak{g}_2, R_2)$ consists of an isometry of $\mathbb{R}^{4,4n+4}$ that defines the equivalence of the representations $\mathfrak{g}_1, \mathfrak{g}_2 \subset \mathfrak{sp}(1, n + 1)$ and sends $R_1$ to $R_2$. For a positive real number $c \in \mathbb{R}$, the symmetric pairs $(\mathfrak{g}, cR)$ and $(\mathfrak{g}, R)$ define diffeomorphic simply connected symmetric spaces and the metrics of these spaces differ by the factor $c$. Hence we may identify $(\mathfrak{g}, cR)$ and $(\mathfrak{g}, R)$.

**Theorem 4.** *Let $(M, h)$ be a non-flat simply connected pseudo-hyper-Kählerian symmetric space of signature $(4, 4n + 4)$ $(n \geqslant 1)$ and $\mathfrak{g} \subset \mathfrak{sp}(1, n + 1)$ its holonomy algebra. Then $n = 2$,*

$$\mathfrak{g} = L' \ltimes \operatorname{Im}\mathbb{H},$$

*there exists a basis $e_1, e_2$ of $\mathbb{H}^2$ such that $L' = \operatorname{span}_{\mathbb{R}}\{e_1, e_2, je_2 + ie_1\}$, the Gram matrix of $g$ with respect to this basis equals to $G = \begin{pmatrix} 1 & -\frac{1}{2}k \\ \frac{1}{2}k & 1 \end{pmatrix}$. The manifold $(M, h)$ is defined by the symmetric pair $(\mathfrak{g}, R)$, where $R$ is defined as in Proposition 4.1 and it is given by $S_{01} = e_1$, $S_{02} = -e_2$ with other elements defining $R$ being zero.*

**Proof.** Since $(M, h)$ is Ricci-flat, its holonomy algebra $\mathfrak{g}$ cannot be reductive [2], hence it is conjugated to a subalgebra of $\mathfrak{sp}(1, n + 1)_{\mathbb{H}p}$. Let $(\mathfrak{g}, R)$ be a symmetric pair. Then the tensor $R$ is given as in Proposition 4.1.

First suppose that $\operatorname{pr}_{\mathbb{H}} \mathfrak{g} \neq 0$. Let $\xi = (a, A, 0, 0) \in \mathfrak{g}$. Then

$$\big[\xi, R(I_r q, I_s q)\big] - R(\xi I_r q, I_s q) - R(I_r q, \xi I_s q) = 0.$$

Taking the projection on $\mathbb{H}$, we get the same equations on $a$, $C_{rs}$ as in [8, Section 4], where it is shown that these equations imply $C_{rs} = 0$. Hence, $R(\mathbb{R}^{4,4n+4}, \mathbb{R}^{4,4n+4}) \neq \mathfrak{g}$ and we obtain a contradiction. Thus, $\operatorname{pr}_{\mathbb{H}} \mathfrak{g} = 0$.

Since $\operatorname{pr}_{\mathbb{H}} \mathfrak{g} = 0$, we get $C_{rs} = B_{rs} = 0$. Let $\xi = (0, 0, Y, 0) \in \mathfrak{g}$ and $X, Z \in \mathbb{H}^n$. The condition $(\xi \cdot R)(X, Z) = 0$ implies $R' = 0$ and $\operatorname{Im} g(Y, Q(X, Z)) = 0$. This shows that $Q = 0$ and $P_0 = 0$. Suppose that $Y \in \mathbb{H}^m$. The condition $(\xi \cdot R)(q, I_s q) = 0$ implies

$$-A_{0s}Y + T_s(Y) - T_0(I_s Y) = 0.$$

Substituting $T_s(Y) = I_s T_0(Y) - A_{0s}(Y)$, we get

$$-2A_{0s}Y + I_s T_0(Y) - T_0(I_s Y) = 0.$$

Replacing $Y$ by $I_s Y$, multiplying the obtained equality by $I_s$, and combining it with the last one, we obtain $A_{0s}Y = 0$. Hence, $A_{0s} = 0$.

Now $R$ is defined only by $S_{rs}$ and $D_{rs}$ and $L$ must be spanned by $S_{01}, S_{02}, S_{03}$. This shows that $\mathfrak{g} = L \ltimes \operatorname{Im}\mathbb{H}$ and $L$ has dimension at most 3. Hence $n = 1$ or 2. If $n = 1$, then either $L = \operatorname{Im}\mathbb{H}$, or $L = \mathbb{C}$. If $n = 2$, then $L = L'$ and $\dim L' = 3$.

Let $Y \in L$ and $\xi = (0, 0, Y, 0) \in \mathfrak{g}$. The condition $(\xi \cdot R)(q, I_s q) = 0$ implies

$$2\operatorname{Im} g(Y, S_{0s}) = \theta_0(I_s Y) - \theta_s(Y). \tag{30}$$

From this equation, (19) and (20), we have

$$\operatorname{Im} g(Y, S_{0s}) = \frac{1}{2}\big(I_1 g(I_s Y, S_{01}) + I_2 g(I_s Y, S_{02}) + I_3 g(I_s Y, S_{03})\big).$$

Using (18), we get

$$\operatorname{Im} g(Y, S_{0s}) = \frac{1}{2}\big(I_1 g(I_s Y, S_{01}) + I_2 g(I_s Y, S_{02}) - I_3 g(I_s Y, S_{01}) I_2 + I_3 g(I_s Y, S_{02}) I_1\big), \tag{31}$$

where $Y \in L$. Substituting in turn $Y = S_{01}$ and $s = 1$; $Y = S_{02}$ and $s = 2$ in the last equation, we obtain the following

$$0 = -k g(S_{01}, S_{02}) + j g(S_{01}, S_{02}) i, \tag{32}$$

$$0 = k g(S_{02}, S_{01}) + i g(S_{02}, S_{01}) j. \tag{33}$$

Taking the conjugation to (33), we get

$$0 = -g(S_{01}, S_{02}) k + j g(S_{01}, S_{02}) i. \tag{34}$$

From (32) and (34), we have $g(S_{01}, S_{02})k - k g(S_{01}, S_{02}) = 0$. It means that $g(S_{01}, S_{02}) = \alpha + \beta k$, $\alpha, \beta \in \mathbb{R}$. Substituting this to (32), we obtain $\alpha = 0$, i.e.

$$g(S_{01}, S_{02}) = \beta k, \quad \beta \in \mathbb{R}. \tag{35}$$

If we take $Y = S_{01}$ and $s = 2$ in (31), then

$$g(S_{01}, S_{01}) = 2\beta. \tag{36}$$

Similarly, $g(S_{02}, S_{02}) = 2\beta$. If $\beta = 0$, then $S_{01} = S_{02} = 0$ and consequently, $S_{rs} = 0$.

Now let $\beta \neq 0$, then $S_{01} \neq 0$ and $S_{02} \neq 0$. Let $n = 1$. If $L = \mathbb{C}e$, then $S_{01}, S_{02}, S_{03} \in \mathbb{C}e$. At the same time $S_{03} = jS_{01} - iS_{02} \in \mathbb{C}e$, from here $S_{01} = 0$ and $\beta = 0$. Further, let $L = \operatorname{Im}\mathbb{H}e$. Since $S_{01} \neq 0$, there exists $a \in \mathbb{H}$ such that $S_{02} = aS_{01}$. Using (35) and (36), we get $a = -\frac{1}{2}k$, i.e. $S_{02} = -\frac{1}{2}kS_{01}$ and $S_{03} = \frac{1}{2}jS_{01}$. The condition $S_{0s} \in \operatorname{Im}\mathbb{H}e$ implies $S_{01} = \gamma i e$, $\gamma \in \mathbb{R}$ (see [8, Section 4]). Therefore, $S_{02} = -\frac{1}{2}\gamma j e$ and $S_{03} = -\frac{1}{2}\gamma k e$. Substituting $Y = je$, $s = 3$ and obtained above $S_{01}, S_{02}, S_{03}$ to (31), we prove that $\gamma = 0$.

Let $n = 2$, then $L = L'$ and we obtain $L' = \{e_1, e_2, je_1 + ie_2\}$ for some basis $(e_1, e_2)$ of $\mathbb{H}^2$. Let us find $S_{rs}$. One may write

$$S_{0s} = a_s e_1 + b_s e_2 + c_s(je_1 + ie_2),$$

where $a_s, b_s, c_s \in \mathbb{R}$. The condition $S_{03} = jS_{01} - iS_{02}$ implies $S_{01} = a_1 e_1$ and $S_{02} = -a_1 e_2$. We may assume that $a_1 = 1$. From the above reasoning it follows that $G = \begin{pmatrix} 2\beta & -\beta k \\ \beta k & 2\beta \end{pmatrix}$ for some $\beta \in \mathbb{R}$, $\beta > 0$. Changing $e_1$, $e_2$ by $\frac{\sqrt{2\beta}}{2\beta}e_1$, $\frac{\sqrt{2\beta}}{2\beta}e_2$, we get $G = \begin{pmatrix} 1 & -\frac{1}{2}k \\ \frac{1}{2}k & 1 \end{pmatrix}$.

Note that we still have possibly non-zero elements $D_{rs}$. Let us consider $p' = p$, $q' = -\frac{g(X,X)}{2}p + X + q$ for some $X \in \mathbb{H}^n$. This defines a new basis $p'$, $e'_1$, $e'_2$, $q'$ of $\mathbb{H}^{1,3}$. With respect to this basis $R$ is given by the elements $S'_{rs}$ and $D'_{rs}$. As we have shown just now, the vectors $e'_1$ and $e'_2$ can be chosen to have the same properties as $e_1$, $e_2$, i.e. we may assume that $S'_{rs}$ remain the same. It holds

$$D'_{rs} = D_{rs} - \theta_s(I_r X) + \theta_r(I_s X) - g(X, S_{rs}) + g(S_{rs}, X).$$

Choosing $r = 0$ in the above equation and using (20), we get

$$D'_{0s} = D_{0s} + 2\theta_0(I_s X) + 2\operatorname{Im} g(S_{0s}, X).$$

In the proof of Theorem 4 (see [8, Section 7]), it has been shown that if we consider the new vectors $\tilde{p} = xp$, $\tilde{q} = xq$, for some $x \in \mathbb{H}$, then $D_{01} = \mu i$, $D_{02} = \lambda j$, where $\mu, \lambda \in \mathbb{R}$ (again, this will not change $S_{rs}$). Now take $X = \frac{\mu}{4}ie_1 + \frac{\mu}{4}je_2$, then

$$D'_{01} = 0, \qquad D'_{02} = (\mu + \lambda)j, \qquad D'_{03} = -(\mu + \lambda)k.$$

Again, by similar reasoning as in [8, Section 7], after some transformation $\tilde{p} = xp$, $\tilde{q} = xq$, $x \in \mathbb{H}$, $x\bar{x} = 1$, we obtain

$$D'_{01} = (\mu + \lambda)i, \qquad D'_{02} = -(\mu + \lambda)j, \qquad D'_{03} = 0.$$

Once more, consider $q' = -\frac{g(X,X)}{2}p + X + q$, where $X = \frac{\mu+\lambda}{4}ie_1 + \frac{\mu+\lambda}{4}je_2$, then $D'_{01} = D'_{02} = 0$. Hence, $D'_{rs} = 0$. This proves the theorem. □

The symmetric spaces from the above theorem and from [8, Theorem 4] coincide with the ones obtained in [21,22].

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