

## Examples of Einstein spacetimes with recurrent null vector fields

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2011 Class. Quantum Grav. 28 175022

(<http://iopscience.iop.org/0264-9381/28/17/175022>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 147.251.82.136

The article was downloaded on 22/09/2011 at 10:03

Please note that [terms and conditions apply](#).

# Examples of Einstein spacetimes with recurrent null vector fields

**Anton S Galaev**

Department of Mathematics and Statistics, Faculty of Science, Masaryk University in Brno,  
Kotlářská 2, 611 37 Brno, Czech Republic

E-mail: [galaev@math.muni.cz](mailto:galaev@math.muni.cz)

Received 30 May 2011

Published 17 August 2011

Online at [stacks.iop.org/CQG/28/175022](http://stacks.iop.org/CQG/28/175022)

## Abstract

The Einstein equation on four-dimensional Lorentzian manifolds admitting recurrent null vector fields is discussed. Several examples of a special form are constructed. The holonomy algebras, Petrov types and the Lie algebras of Killing vector fields of the obtained metrics are found.

PACS numbers: 04.20.Jb, 04.40.Nr

## 1. Introduction

We study the Einstein equation on spacetimes admitting parallel distributions of null lines and construct several special examples.

Let  $(M, g)$  be a four-dimensional Lorentzian manifold admitting a parallel distribution of null lines (i.e.  $(M, g)$  is a *Walker manifold*). This condition holds if and only if the holonomy group of  $(M, g)$  is contained in the subgroup  $\text{Sim}(2) \subset \text{O}(1, 3)$  preserving a null line in the Minkowski space  $\mathbb{R}^{1,3}$ . Equivalently,  $(M, g)$  admits a recurrent null vector field in a neighborhood of each point, and locally there exist coordinates  $v, x^1, x^2, u$  (so-called *Walker coordinates*) such that the metric  $g$  has the form

$$g = 2 dv du + h + 2A du + H \cdot (du)^2, \quad (1)$$

where  $h = h_{ij}(x^1, x^2, u) dx^i dx^j$  is a  $u$ -dependent family of Riemannian metrics,  $A = A_i(x^1, x^2, u) dx^i$  is a  $u$ -dependent family of one-forms and  $H$  is a local function on  $M$  [18]. The vector field  $\partial_v = \frac{\partial}{\partial v}$  is null and recurrent, and it defines the parallel distribution of null lines. We will also write  $x^1 = x, x^2 = y$ .

A Lorentzian manifold  $(M, g)$  is called an *Einstein manifold* if  $g$  satisfies the equation

$$\text{Ric} = \Lambda g, \quad \Lambda \in \mathbb{R}, \quad (2)$$

where  $\text{Ric}$  is the Ricci tensor of the metric  $g$ , i.e.  $\text{Ric}_{ab} = R^c_{acb}$ , where  $R$  is the curvature tensor of the metric  $g$ . The number  $\Lambda \in \mathbb{R}$  is called *the cosmological constant*. If  $\Lambda = 0$ ,

i.e.  $\text{Ric} = 0$ , then the manifold is called *Ricci-flat or vacuum Einstein*. We consider the case  $\Lambda \neq 0$ .

The Walker manifolds are of particular type of Kundt spaces [2, 4]. Recently Gibbons and Pope [8] considered the Einstein equation on Walker manifolds of arbitrary dimension.

A special example of metric (1) is the metric of a pp-wave. It is given by  $h = (dx)^2 + (dy)^2$ ,  $A = 0$  and  $H$  independent of  $v$ . If such a metric is Einstein, then it is vacuum Einstein, and this happens whenever  $(\partial_x^2 + \partial_y^2)H = 0$ . For the plane waves ( $H = A_{ij}(u)x^i x^j$ ), sometimes it is useful to rewrite the metric in the Rosen coordinates,  $g = 2 dv du + h$ , where  $h = C_{ij}(u)x^i x^j$  is a family of flat metrics, see e.g. [1, 15, 17]. The examples of the Einstein metrics that we construct have a similar structure.

Vacuum Einstein Walker metrics in dimension 4 are found in [12]. After a proper change of coordinates, they are given by  $h = (dx)^2 + (dy)^2$ ,  $A_2 = 0$ ,  $H = -(\partial_x A_1)v + H_0$ , where  $A_1$  is a harmonic function and  $H_0$  can be found from a Poisson equation. In [5] several examples of such metrics are rewritten in new coordinates such that  $A = 0$  and  $h$  is a  $u$ -family of flat metrics on  $\mathbb{R}^2$ .

In [13], all four-dimensional Einstein Walker metrics with  $\Lambda \neq 0$  are described. After a proper choice of the coordinates,  $h$  becomes independent of the  $u$  metric of constant curvature. Next,  $A = W dz + \bar{W} \bar{z}$ ,  $W = i \partial_z L$ , where  $z = x + iy$ ,  $L$  is the  $\mathbb{R}$ -valued function given by

$$L = 2 \text{Re} \left( \phi \partial_z (\ln P_0) - \frac{1}{2} \partial_z \phi \right), \quad 2P_0^2 = \left( 1 + \frac{\Lambda}{|\Lambda|} z \bar{z} \right)^2, \quad (3)$$

where  $\phi = \phi(z, u)$  is an arbitrary function holomorphic in  $z$  and smooth in  $u$ . Finally,  $H = \Lambda^2 v + H_0$ , where the function  $H_0 = H_0(z, \bar{z}, u)$  can be found in a similar way.

An example of the Einstein Walker metric with  $\Lambda \neq 0$  is given in [7]. After a change of the coordinates in [8], this metric is given by  $A = 0$  and  $h$  is independent of  $u$ . In [3], the universality of the metric from [7] and more generally of Einstein metrics with  $\text{Sim}(2)$ -holonomy is proved.

The aim of this paper is to construct the examples of Einstein Walker metrics with  $\Lambda \neq 0$  such that  $A = 0$  and  $h$  depends on  $u$ . The solutions from [13] are not useful for constructing examples of such form, since ‘simple’ functions  $\phi(z, u)$  define complicated forms  $A$ . First we find Walker metrics with  $h$  independent of  $u$  and  $A \neq 0$ , and then we change the coordinates in such a way that  $A = 0$ . The constructed examples can be useful according to [2–4, 8]. Similar examples can be constructed in dimension 5; this dimension is discussed e.g. in [8, 9].

## 2. Coordinate transformations and reduction of the Einstein equation

The Einstein equation (2) for the Walker metric (1) in an arbitrary dimension is written down in [8]. It is shown that  $h$  is a family of Einstein Riemannian metrics with the cosmological constant  $\Lambda$ ,  $H = \Lambda v^2 + H_1 v + H_0$ ,  $\partial_v H_1 = \partial_v H_0 = 0$ , and there are three additional equations.

The Walker coordinates are not defined canonically and any other Walker coordinates  $\tilde{v}$ ,  $\tilde{x}^1$ ,  $\tilde{x}^2$ ,  $\tilde{u}$  such that  $\partial_{\tilde{v}} = \partial_v$  are given by the transformation

$$\tilde{v} = v + f(x^1, x^2, u), \quad \tilde{x}^i = \tilde{x}^i(x^1, x^2, u), \quad \tilde{u} = u + c; \quad (4)$$

see [5, 17]. If metric (1) is Einstein with  $\Lambda \neq 0$ , then there exist Walker coordinates such that  $A = 0$  and  $H_1 = 0$  [5]. Then the Einstein equation takes the form

$$\Delta H_0 + \frac{1}{2} h^{ij} \ddot{h}_{ij} = 0, \quad \nabla^j \dot{h}_{ij} = 0, \quad h^{ij} \dot{h}_{ij} = 0, \quad \text{Ric}_{ij} = \Lambda h_{ij}, \quad (5)$$

where  $\dot{h}_{ij} = \partial_u h_{ij}$ . Thus, we obtain two equations on the family of Riemannian Einstein metrics and the Poisson equation for the function  $H_0$ . Of course, a solution gives the constant

family,  $\dot{h}_{ij} = 0$  and a harmonic  $H_0$ , but it is more interesting to find solutions with  $\dot{h}_{ij} \neq 0$ . This will be done below.

In order to find the coordinates as in [5], we start with any Walker coordinates and consider two transformations. First consider the transformation  $v \mapsto v + \frac{1}{2\Lambda} H_1$ , and after that  $H_1 = 0$ . Then consider the transformation  $x^i \mapsto \tilde{x}^i(x^1, x^2, u)$  such that the inverse transformation satisfies the system of ordinary differential equations

$$\frac{dx^i(u)}{du} = W^i(x^1(u), x^2(u), u),$$

where  $W^i = -A_j h^{ij}$ , with the initial conditions  $x^i(u_0) = \tilde{x}^i$ . The solution can then be written as  $x^i = x^i(\tilde{x}^1, \tilde{x}^2, u)$ .

Note that in dimension 2 (and 3) any Einstein Riemannian metric has constant sectional curvature; hence, any such metrics with the same  $\Lambda$  are locally isometric, and the coordinates can be chosen in such a way that  $\partial_u h = 0$ . As in [13], it is not hard to show that if  $\Lambda > 0$ , then we may choose  $h = (dx)^2 + \sin^2 x (dy)^2$ ,  $H = \Lambda v^2 + H_0$  and the Einstein equation is reduced to

$$\Delta_{S^2} f = -2f, \quad \Delta_{S^2} H_0 = 2\Lambda \left( 2f^2 - (\partial_x f)^2 + \frac{(\partial_y f)^2}{\sin^2 x} \right), \quad \Delta_{S^2} = \partial_x^2 + \frac{\partial_y^2}{\sin^2 x} + \cot x \partial_x. \tag{6}$$

The function  $f$  defines the one-form  $A$ :  $A = -\frac{\partial_y f}{\sin x} dx + \sin x \partial_x f dy$ . Similarly, if  $\Lambda < 0$ , then we chose  $h = \frac{1}{-\Lambda x^2} ((dx)^2 + (dy)^2)$  and obtain

$$\Delta_{L^2} f = 2f, \quad \Delta_{L^2} H_0 = -4\Lambda f^2 - 2\Lambda x^2 ((\partial_x f)^2 + (\partial_y f)^2), \quad \Delta_{L^2} = x^2 (\partial_x^2 + \partial_y^2), \tag{7}$$

and  $A = -\partial_y f dx + \partial_x f dy$ . Thus, in order to find partial solutions of (5), it is convenient to solve one of the above equations for  $f$  and then, changing the coordinates, to get rid of  $A$ . Note that *after such a change,  $h$  does not depend on  $u$  if and only if  $A$  is a Killing form for  $g$*  [8]. For  $\Lambda > 0$ , this happens if and only if  $f = c_1(u) \sin x \sin y + c_2(u) \sin x \cos y + c_3(u) \cos x$ ; for  $\Lambda < 0$ , this happens if and only if  $f = c_1(u) \frac{1}{x} + c_2(u) \frac{y}{x} + c_3(u) \frac{x^2+y^2}{x}$ . The functions  $\phi(z, u) = c(u)$ ,  $c(u)z$  and  $c(u)z^2$  from (3) define a Killing form  $A$ , see [5], and  $A$  becomes complicated for other  $\phi$ .

Let  $g$  now be a Walker Einstein metric (1) with  $A = 0$ , and let  $\tilde{h}(u)$  be a family of Riemannian metrics such that  $h(u_0) = \tilde{h}(u_0)$  for some  $u_0 \in \mathbb{R}$ . Then *the metric  $g$  is isometric to the metric  $\tilde{g} = 2dv du + \tilde{h} + \tilde{H}(du)^2$  with some  $\tilde{H}$  if and only if  $h = \tilde{h}$  for all  $u$* . Indeed, if the metrics are isometric, then one can be taken to another using transformation (4). Since  $h(u_0) = \tilde{h}(u_0)$ , we may assume that  $\tilde{x}^k(x^1, x^2, u_0) = x^k$ . Note that  $A_i = \frac{\partial \tilde{x}^j}{\partial x^i} (\tilde{A}_j + \tilde{h}_{jk} \frac{\partial \tilde{x}^k}{\partial u})$ ; hence,  $\frac{\partial \tilde{x}^k}{\partial u} = 0$ . We conclude that  $h = \tilde{h}$ . The converse statement is trivial.

### 3. Curvature, holonomy and Petrov type

Let  $g$  be an Einstein metric of the form (1) with  $\Lambda \neq 0$ ,  $A = 0$  and  $H = \Lambda v^2 + H_0$ , i.e.  $H_1 = 0$ . Consider the vector fields  $p = \partial_v$ ,  $q = \partial_u - \frac{1}{2} H \partial_v$  and the distribution  $E$  generated by  $\partial_x$  and  $\partial_y$ . We will use the identification  $\Lambda^2 \mathbb{R}^{1,3} \simeq \mathfrak{so}(1, 3)$ . In [5] it is shown that the curvature tensor  $R$  of the metric  $g$  is given by

$$R(p, q) = \Lambda p \wedge q, \quad R(X, Y) = \Lambda X \wedge Y, \quad R(X, q) = -p \wedge T(X), \quad R(p, X) = 0,$$

where  $X, Y$  are the sections of  $E$ , and  $T(X) = -R(X, q)q$  is a symmetric endomorphism of  $E$ . It holds  $T_i^j = -R_{4i4}^j$ . Obviously, the metric  $g$  is indecomposable if and only if  $T \neq 0$ . In

this case, the holonomy algebra at any point  $m \in M$  equals  $\mathbb{R}p_m \wedge q_m + \mathfrak{so}(E_m) + p_m \wedge E_m$  and coincides with the maximal subalgebra  $\mathfrak{sim}(2) \subset \mathfrak{so}(1, 3)$  preserving the null line  $\mathbb{R}p_m$ . If  $T$  is identically zero, then the manifold is decomposable and the holonomy algebra coincides with  $\mathbb{R}p_m \wedge q_m \oplus \mathfrak{so}(E_m)$ . The holonomy algebras of the Einstein Walker metrics are also found in the papers [6, 10, 16]. Remark that the manifolds under the consideration never admit (even locally) any null parallel vector field.

For the Weyl tensor, we obtain

$$\begin{aligned} W(p, q) &= \frac{\Lambda}{3} p \wedge q, & W(p, X) &= -\frac{2\Lambda}{3} p \wedge X, \\ W(X, Y) &= \frac{\Lambda}{3} X \wedge Y, & W(X, q) &= -\frac{2\Lambda}{3} X \wedge q - p \wedge T(X). \end{aligned}$$

In [10, 11], it is shown that the Petrov type of the metric  $g$  is either II or D (and it may change from point to point); in particular, the manifolds under the consideration are *algebraically special* in the sense of the Petrov classification [14]. Using the Bel criteria and the above expression for  $W$ , it is easy to see that  $g$  has type II at a point  $m \in M$  if and only if  $T_m \neq 0$ , and  $g$  has type D at a point  $m \in M$  if and only if  $T_m = 0$ . Since  $T_m$  is symmetric and trace free, it is either zero or it has rank 2. Hence,  $T_m = 0$  if and only if  $\det T_m = 0$ .

#### 4. Examples

*Case  $\Lambda < 0$ .* We consider a partial solution of equation (7). This gives an Einstein metric. Then we find new coordinates such that  $A = 0$  and  $h$  may depend on  $u$ .

**Example 1.** Let  $\partial_y f = 0$ ; then  $f$  satisfies  $x^2 \partial_x^2 f - 2f = 0$ . Hence,  $f = \frac{c_1(u)}{x} + x^2 c_2(u)$ . The function  $\frac{c_1(u)}{x}$  defines a Killing form, and we take  $f = c(u)x^2$ ; then  $A = 2xc(u) dy$ , and we chose  $H_0 = -\Lambda x^4 c^2(u)$ . To get rid of  $A$ , we need to solve the system of equations

$$\frac{dx(u)}{du} = 0, \quad \frac{dy(u)}{du} = 2\Lambda c(u)x^3(u).$$

Imposing the initial conditions  $x(0) = \tilde{x}$  and  $y(0) = \tilde{y}$ , we obtain that the inverse transformation to the required one has the form

$$v = \tilde{v}, \quad x = \tilde{x}, \quad y = \tilde{y} + 2\Lambda b(u)\tilde{x}^3, \quad u = \tilde{u},$$

where  $b(u)$  is the function such that  $\frac{db(u)}{du} = c(u)$  and  $b(0) = 0$ . With respect to the obtained coordinates, we get

$$g = 2 dv du + h(u) + (\Lambda v^2 + 3\Lambda x^4 c^2(u))(du)^2,$$

$$h(u) = \frac{1}{-\Lambda \cdot x^2} ((36\Lambda^2 b^2(u)x^4 + 1)(dx)^2 + 12\Lambda b(u)x^2 dx dy + (dy)^2).$$

Let  $c(u) \equiv 1$ ; then  $b(u) = u$ . It holds  $\det T = -9\Lambda^4 x^4 (x^4 + v^2)$ . This shows that  $\det T_m = 0$  ( $m = (v, x, y, u)$ ) if and only if  $v = 0$ . In particular, the metric is indecomposable. The metric  $g$  is of Petrov type D on the set  $\{(0, x, y, u)\}$ , and it is of type II on its complement.

If  $b(u) = u$ , then the Lie algebra of Killing vector fields of the obtained metric is spanned by the vector fields  $\partial_y, 2v\partial_v + x\partial_x + y\partial_y - 2u\partial_u, \partial_u - 2\Lambda x^3 \partial_y$ .

**Example 2.** The functions  $f = x^2 y$  and  $H_0 = -\Lambda x^4 y$  are partial solutions of (7). Then,  $A = -x^2 dx du + 2xy dy$ . Consider the transformation with the inverse one given by

$$v = \tilde{v}, \quad x = \tilde{x}(1 + 3\Lambda \tilde{u} \tilde{x}^3)^{-\frac{1}{3}}, \quad y = \tilde{y}(1 + 3\Lambda \tilde{u} \tilde{x}^3)^{\frac{2}{3}}, \quad u = \tilde{u}.$$

With respect to the obtained coordinates, we get

$$g = 2 dv du + h(u) + \Lambda \left( v^2 + 3x^4 y^2 + \frac{x^6}{\rho^2} \right) (du)^2, \quad \rho = 1 + 3\Lambda u x^3,$$

$$h(u) = \frac{1}{-\Lambda} \left( \left( 36\Lambda^2 x^2 y^2 u^2 + \frac{1}{x^2 \rho^2} \right) (dx)^2 + 12\Lambda \rho y u dx dy + \frac{\rho^2}{x^2} (dy)^2 \right).$$

It can be checked that  $\det T < 0$  everywhere; hence, the metric  $g$  is indecomposable and it is of Petrov type II everywhere.

The Lie algebra of Killing vector fields of the obtained metric is spanned by the vector fields  $3v\partial_v + x\partial_x + y\partial_y - 3u\partial_u$  and  $\Lambda x^4 \partial_x - 2\Lambda x^3 y \partial_y + \partial_u$ .

The above two metrics are not isometric after any change of the functions  $H_0$ ; see section 2.

Case  $\Lambda > 0$ . Assuming  $\partial_y f = 0$ , we obtain the solution

$$f = c_1(u) \cos x + c_2(u) \left( \ln \left( \tan \frac{x}{2} \right) \cos x + 1 \right)$$

of equation (6). Obviously,  $yf$  is a solution of equation (6) as well. The function  $c_1(u) \cos x$  defines a Killing form  $A$ , and we omit it.

**Example 3.** The function  $f = \ln \left( \tan \frac{x}{2} \right) \cos x + 1$  is a solution of the first equation from (6). We obtain  $A = (\cos x - \ln \left( \cot \frac{x}{2} \right) \sin^2 x) dy$ . Consider the transformation

$$\tilde{v} = v, \quad \tilde{x} = x, \quad \tilde{y} = y - \Lambda u \left( \ln \left( \tan \frac{x}{2} \right) - \frac{\cos x}{\sin^2 x} \right), \quad \tilde{u} = u.$$

With respect to the obtained coordinates, we get

$$g = 2 dv du + h(u) + (\Lambda v^2 + \tilde{H}_0)(du)^2,$$

$$h(u) = \left( \frac{1}{\Lambda} + \frac{4\Lambda u^2}{\sin^4 x} \right) (dx)^2 + \frac{4u}{\sin x} dx dy + \frac{\sin^2 x}{\Lambda} (dy)^2,$$

where  $\tilde{H}_0$  satisfies  $\Delta_h \tilde{H}_0 = -\frac{1}{2} h^{ij} \tilde{h}_{ij}$ . An example of such  $\tilde{H}_0$  is

$$\tilde{H}_0 = -\Lambda \left( \frac{1}{\sin^2 x} + \ln^2 \left( \cot \frac{x}{2} \right) \right).$$

Coming back to the initial coordinates, we obtain  $H_0 = \Lambda \cdot (\ln \left( \tan \frac{x}{2} \right) \cos x + 1)$ . We have

$$\det T = -\frac{\Lambda^4}{\sin^4 x} \left( v^2 + \left( \ln \left( \cot \frac{x}{2} \right) \cos x - 1 \right)^2 \right).$$

Hence,  $g$  is of Petrov type D on the set  $\{(0, x, y, u) \mid \ln \left( \cot \frac{x}{2} \right) \cos x - 1 = 0\}$ , and it is of type II on the complement to this set. The metric is indecomposable.

The Lie algebra of the Killing vector fields of the obtained metric is spanned by the vector fields  $\partial_y$  and  $\partial_u + \Lambda \left( \frac{\cos x}{\sin^2 x} - \ln \left( \tan \frac{x}{2} \right) \right) \partial_y$ .

**Example 4.** The functions  $f = y \cos x$  and  $H_0 = \Lambda \cdot (-y^2 \cos^2 x + \ln \left( \tan \frac{x}{2} \right))$  are partial solutions of (6). We obtain  $A = -\cot x dx - y \sin^2 x dy$ . Consider the transformation

$$\tilde{v} = v, \quad \tilde{x} = \arccos(e^{\Lambda u} \cos x), \quad \tilde{y} = y e^{-\Lambda u}, \quad \tilde{u} = u.$$

With respect to the obtained coordinates, we get

$$g = 2 dv du + h(u) + \Lambda \left( v^2 - y^2 e^{-2\Lambda u} + \frac{1}{2} \ln \frac{1-\rho}{1+\rho} \right) (du)^2, \quad \rho = e^{-\Lambda u} \cos x,$$

$$h(u) = \frac{e^{-2\Lambda u} \sin^2 x}{\Lambda \cdot (1 - \rho^2)} (dx)^2 + \frac{e^{2\Lambda u} (1 - \rho^2)}{\Lambda} (dy)^2.$$

It holds  $\det T = -\frac{\Lambda^4 \cdot (4y^2 \cos^2 x + (\rho + 2v)^2)}{4(1 - \rho^2)^2}$ . Hence, the metric is indecomposable and  $T$  vanishes at a point  $(v, x, y, u)$  if and only if  $y \cos x = 0$  and  $e^{-\Lambda u} \cos x + 2v = 0$ . Consequently, the metric  $g$  is of Petrov type D at a point  $(v, x, y, u)$  if and only if  $x = \frac{\pi}{2}$  and  $v = 0$ , or  $y = 0$  and  $\cos x = -2v e^{\Lambda u}$ . It is of Petrov type II at other points.

The Lie algebra of the Killing vector fields of the obtained metric is spanned by the vector field  $-\Lambda \cot x \partial_x - \Lambda y \partial_y + \partial_u$ .

The metrics of the last two examples are not isometric after any change of  $H_0$ .

## Acknowledgments

I am grateful to D V Alekseevsky for helpful suggestions and Thomas Leistner for useful discussions. Most of the calculations are done using Maple 12. I am grateful to Ian Anderson for tutorials on Maple. The final form of the paper is due to helpful remarks from the referee. The work was supported by the grant CZ.1.07/2.3.00/20.0003.

## References

- [1] Blau M, Figueroa-O'Farrill J and Papadopoulos G 2002 Penrose limits, supergravity and brane dynamics *Class. Quantum Grav.* **19** 4753–805
- [2] Brannlund J, Coley A and Hervik S 2008 Supersymmetry, holonomy and Kundt spacetimes *Class. Quantum Grav.* **25** 195007
- [3] Coley A, Gibbons G W, Hervik S and Pope C N 2008 Metrics with vanishing quantum corrections *Class. Quantum Grav.* **25** 145017
- [4] Coley A, Hervik S, Papadopoulos G and Pelavas N 2009 Kundt spacetimes *Class. Quantum Grav.* **26** 105016
- [5] Galaev A S and Leistner T 2010 On the local structure of Lorentzian Einstein manifolds with parallel distribution of null lines *Class. Quantum Grav.* **27** 225003
- [6] Galaev A S 2010 Holonomy of Einstein Lorentzian manifolds *Class. Quantum Grav.* **27** 075008
- [7] Ghanam R and Thompson G 2001 Two special metrics with  $R_{14}$ -type holonomy *Class. Quantum Grav.* **18** 2007–14
- [8] Gibbons G W and Pope C N 2008 Time-dependent multi-centre solutions from new metrics with holonomy  $\text{Sim}(n-2)$  *Class. Quantum Grav.* **25** 125015
- [9] Grover J *et al* 2009 Gauduchon–Tod structures, Sim holonomy and de Sitter supergravity *J. High Energy Phys.* JHEP07(2009)069
- [10] Hall G S and Lonie D P 2000 Holonomy groups and spacetimes *Class. Quantum Grav.* **17** 1369–82
- [11] Hall G S 2004 *Symmetries and Curvature Structure in General Relativity* (Singapore: World Scientific)
- [12] Kerr R P and Goldberg J N 1961 Einstein spaces with four-parameter holonomy groups *J. Math. Phys.* **2** 332–6
- [13] Lewandowski J 1992 Reduced holonomy group and Einstein equations with a cosmological constant *Class. Quantum Grav.* **9** L147–51
- [14] Petrov A Z 1969 *Einstein Spaces* (Oxford: Pergamon) xiii+411 pp
- [15] Ivashchuk V D 2003 Generalized pp-wave solutions on products of Ricci-flat spaces *Gravit. Cosmol.* **9** 146–52
- [16] Schell J F 1961 Classification of four-dimensional Riemannian spaces *J. Math. Phys.* **2** 202–6
- [17] Schimming R 1974 Riemannsche Raume mit ebenfrontiger und mit ebener *Symmetrie Math. Nachr.* **59** 129–62
- [18] Walker A G 1949 On parallel fields of partially null vector spaces *Q. J. Math. (Oxford Ser.)* **20** 135–45