

Note on the holonomy groups of pseudo-Riemannian manifolds

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For an arbitrary subalgebra $\mathfrak{h} \subset \mathfrak{so}(r, s)$, a polynomial pseudo-Riemannian metric of signature $(r+2, s+2)$ is constructed, the holonomy algebra of this metric contains \mathfrak{h} as a subalgebra. This result shows the essential distinction of the holonomy algebras of pseudo-Riemannian manifolds of index bigger or equal to 2 from the holonomy algebras of Riemannian and Lorentzian manifolds.

1. Introduction. The holonomy group of a linear connection on an n -dimensional manifold is contained in the Lie group $GL(n, \mathbb{R})$ and it represents an important invariant of the connection. Hano and Ozeki [6] showed that any connected linear Lie group $G \subset GL(n, \mathbb{R})$ may be realized as the holonomy group of a space with a linear connection. This connection, as a rule, are of non-zero torsion. On the contrary, the absence of the torsion imposes some algebraic condition on the holonomy group. Berger used this condition in order to get the classification of the connected holonomy groups of Riemannian manifolds and of connected irreducible holonomy groups of the spaces with torsion-free linear connections [3]. Later the Berger lists were corrected. The classification of the connected irreducible holonomy groups of torsion-free linear connections is obtained in [11]. The most important result turned out to be the classification of the connected holonomy groups of Riemannian manifolds, this result has many applications in geometry and theoretical physics [1, 2, 8, 7]. Lately the interest to the pseudo-Riemannian manifolds appears. Recently the classification of the connected holonomy groups of Lorentzian manifolds was obtained [4, 10]. Note that the study of the connected holonomy groups is equivalent to the study of the holonomy algebras, i.e. the corresponding Lie algebras. Consider a Lorentzian manifold of dimension $n+2 \geq 4$. Its holonomy algebra is contained in the Lorentzian Lie algebra $\mathfrak{so}(1, n+1)$. It is enough to consider the holonomy algebras contained in the maximal algebra preserving an isotropic line in the Minkowski space, $\mathfrak{g} \subset \mathfrak{so}(1, n+1)_\ell = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n$. With the algebra \mathfrak{g} its projection \mathfrak{h} to $\mathfrak{so}(n)$ is associated. The key fact is that $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a Riemannian manifold.

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Note that the maximal subalgebra of the pseudo-orthogonal Lie algebra $\mathfrak{so}(r+2, s+2)$, $r+s \geq 2$, preserving a two-dimensional isotropic subspace of the pseudo-Euclidean space $\mathbb{R}^{r+2, s+2}$ has the form $(\mathfrak{gl}(2, \mathbb{R}) \oplus \mathfrak{so}(r, s)) \ltimes (\mathbb{R}^2 \otimes \mathbb{R}^{r, s} \ltimes \mathbb{R})$. In this note, for any subalgebra $\mathfrak{h} \subset \mathfrak{so}(r, s)$ a polynomial pseudo-Riemannian metric of signature $(r+2, s+2)$ is constructed, the holonomy algebra of this metric equals $\mathfrak{h} \ltimes (\mathbb{R}^2 \otimes \mathbb{R}^{r, s} \ltimes \mathbb{R})$. It turns out that the holonomy algebra of a pseudo-Riemannian manifold of signature $(r+2, s+2)$, i.e. of index bigger or equal to 2, can depend on an arbitrary subalgebra $\mathfrak{h} \subset \mathfrak{so}(r, s)$. This indicates the non-visibility of the holonomy algebras of pseudo-Riemannian manifolds of index bigger or equal to 2. In particular, the holonomy algebra of a pseudo-Riemannian manifold of signature $(2, n+2)$, $n \geq 2$, may depend on an arbitrary subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$; this shows the fundamental difference from the case of Lorentzian manifolds. In some sense, the obtained result is analogous to the result from [6]. Other results on the holonomy groups of pseudo-Riemannian manifolds can be found in the review [5].

2. Results. Consider the pseudo-Euclidean space $\mathbb{R}^{r+2, s+2}$ with the pseudo-Euclidean metric η of signature $(r+2, s+2)$, $r+s = n \geq 2$. Let us fix a basis $p_1, p_2, e_1, \dots, e_n, q_1, q_2$ of the space $\mathbb{R}^{r+2, s+2}$ such that η has the following non-zero values:

$$\eta(p_1, q_1) = \eta(p_2, q_2) = 1, \quad \eta(e_i, e_i) = \epsilon_i, \quad \epsilon_1 = \dots = \epsilon_r = -1, \quad \epsilon_{r+1} = \dots = \epsilon_n = 1.$$

Consider the subalgebra $\mathfrak{so}(r+2, s+2)_{\langle p_1, p_2 \rangle} \subset \mathfrak{so}(r+2, s+2)$ preserving the isotropic subspace $\text{span}\{p_1, p_2\} \subset \mathbb{R}^{r+2, s+2}$. This Lie algebra has the following matrix form:

$$\left\{ \left(\begin{array}{cc|cc} B & -(E_{r,s}X)^t & 0 & -c \\ 0 & -(E_{r,s}Y)^t & c & 0 \\ \hline 0 & A & X & Y \\ 0 & 0 & -B^t & 0 \end{array} \right) \middle| \begin{array}{l} B \in \mathfrak{gl}(2, \mathbb{R}), \quad A \in \mathfrak{so}(r, s), \\ X, Y \in \mathbb{R}^{r,s}, \quad c \in \mathbb{R} \end{array} \right\},$$

where $E_{r,s} = \text{diag}(\epsilon_1, \dots, \epsilon_n)$. We get the decomposition

$$\mathfrak{so}(r+2, s+2)_{\langle p_1, p_2 \rangle} = (\mathfrak{gl}(2, \mathbb{R}) \oplus \mathfrak{so}(r, s)) \ltimes (\mathbb{R}^2 \otimes \mathbb{R}^{r,s} \ltimes \mathbb{R}),$$

where the symbol \ltimes indicates that on the right side from it is an ideal.

For an arbitrary subalgebra $\mathfrak{h} \subset \mathfrak{so}(r, s)$ we define the subalgebra

$$\mathfrak{g}^{\mathfrak{h}} = \left\{ \left(\begin{array}{cc|cc} 0 & 0 & -(E_{r,s}X)^t & 0 & -c \\ 0 & 0 & -(E_{r,s}Y)^t & c & 0 \\ \hline 0 & 0 & A & X & Y \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \middle| A \in \mathfrak{h}, X, Y \in \mathbb{R}^{r,s}, c \in \mathbb{R} \right\}$$

of the Lie algebra $\mathfrak{so}(2, n+2)_{\langle p_1, p_2 \rangle}$. We denote an element of the Lie algebra $\mathfrak{g}^{\mathfrak{h}}$ by (A, X, Y, c) . The non-zero Lie brackets are the following:

$$[(A, 0, 0, 0), (A_1, X, Y, 0)] = ([A, A_1], AX, AY, 0), \quad (2.1)$$

$$[(0, X, 0, 0), (0, 0, Y, 0)] = (0, 0, 0, \eta(X, Y)). \quad (2.2)$$

We get the decomposition

$$\mathfrak{g}^{\mathfrak{h}} = \mathfrak{h} \ltimes (\mathbb{R}^2 \otimes \mathbb{R}^{r,s} \ltimes \mathbb{R}).$$

THEOREM 1. *For any subalgebra $\mathfrak{h} \subset \mathfrak{so}(r, s)$, the algebra $\mathfrak{g}^{\mathfrak{h}}$ is the holonomy algebra of a pseudo-Riemannian manifold of signature $(r + 2, s + 2)$.*

Proof. For an arbitrary subalgebra $\mathfrak{h} \subset \mathfrak{so}(r, s)$ we construct a pseudo-Riemannian metric on the space \mathbb{R}^{r+s+4} and show that the holonomy algebra of this metric at the point 0 coincides with $\mathfrak{g}^{\mathfrak{h}}$.

Construction of the metric. We fix elements $B_1, \dots, B_N \in \mathfrak{h}$, generating the Lie algebra \mathfrak{h} . Consider the matrices $(B_{j\alpha}^i)_{i,j=1}^n$ of these elements with respect to the basis e_1, \dots, e_n of the space $\mathbb{R}^{r,s}$. The condition $B_\alpha \in \mathfrak{h} \subset \mathfrak{so}(r, s)$ implies

$$\epsilon_j B_{j\alpha}^i = -\epsilon_i B_{i\alpha}^j.$$

Let $v, z, x^1, \dots, x^n, u, w$ be the standard coordinates on $M = \mathbb{R}^{r+s+4}$. Consider the pseudo-Riemannian metric

$$g = 2dvdu + 2dzdw + \sum_{i=1}^n \epsilon_i (dx^i)^2 + \sum_{i=1}^n 2A_i dx^i dw + f(du)^2 + f(dw)^2,$$

where

$$A_i = \sum_{\alpha=1}^N \sum_{j=1}^n A_{ij\alpha} x^j u^\alpha, \quad A_{ij\alpha} = -\epsilon_i B_{j\alpha}^i, \quad f = \sum_{i=1}^n (x^i)^2.$$

The metric g is of signature $(r + 2, s + 2)$. Let \mathfrak{hol} be the holonomy algebra of this metric at the point 0. Consider the basis

$$p_1 = (\partial_v)_0, p_2 = (\partial_z)_0, e_1 = (\partial_1)_0, \dots, e_n = (\partial_n)_0, q_1 = (\partial_u)_0, q_2 = (\partial_w)_0$$

of the tangent space T_0M . We get $\eta = g_0$, this allows us to identify (T_0M, g_0) with $(\mathbb{R}^{r+2, s+2}, \eta)$ and the holonomy algebra \mathfrak{hol} with a subalgebra of $\mathfrak{so}(r + 2, s + 2)$. ■

Computation of the holonomy algebra. The constructed metric is analytical. From the proof of theorem 9.2 from [9] it follows that \mathfrak{hol} is generated by the elements of the form

$$\nabla_{\partial_{a_\alpha}} \cdots \nabla_{\partial_{a_1}} R(\partial_a, \partial_b)(0) \in \mathfrak{so}(T_0M, g_0) = \mathfrak{so}(r + 2, s + 2), \quad \alpha = 0, 1, 2, \dots,$$

where ∇ is the Levi-Civita connection defined by the metric g and R is the curvature tensor. The indices a, b, c will run through all coordinates on M , the indices i, j, k will take the values $1, \dots, n$. The Levi-Civita connection is determined by its Christoffel symbols Γ_{bc}^a , $\nabla_{\partial_b} \partial_c = \sum_a \Gamma_{bc}^a \partial_a$, which can be found using the formula

$$\Gamma_{bc}^a = \Gamma_{cb}^a = \sum_d g^{ad} (\partial_c g_{bd} + \partial_b g_{cd} - \partial_d g_{bc}),$$

where (g^{ab}) is the inverse matrix to the matrix (g_{ab}) of the metric g . The components of the curvature tensor are defined by the equality $R(\partial_a, \partial_b)\partial_c = \sum_d R_{cab}^d \partial_d$ and can be found in the following way:

$$R_{cab}^d = \partial_a \Gamma_{bc}^d - \partial_b \Gamma_{ac}^d + \sum_e (\Gamma_{bc}^e \Gamma_{ae}^d - \Gamma_{ac}^e \Gamma_{be}^d).$$

For the matrix (g^{ab}) we get

$$(g^{ab}) = \begin{pmatrix} -fE_2 & C & E_2 \\ C^t & E_{r,s} & 0 \\ E_2 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & \cdots & 0 \\ -\epsilon_1 A_1 & \cdots & -\epsilon_n A_n \end{pmatrix}.$$

In order to find the holonomy algebra we will need only the following Christoffel symbols:

$$\Gamma_{jk}^i = 0, \quad \Gamma_{ju}^i = 0, \quad \Gamma_{jw}^i = \sum_{\alpha=1}^N B_{j\alpha}^i u^\alpha, \quad (2.3)$$

$$\Gamma_{1b}^a = \Gamma_{2b}^a = 0, \quad \Gamma_{ia}^u = \Gamma_{ia}^w = 0, \quad (2.4)$$

$$\Gamma_{ij}^1 = \Gamma_{ij}^2 = 0, \quad \Gamma_{iu}^1 = x^i, \quad \Gamma_{iw}^2 = x^i + \sum_k \epsilon_k A_k (\partial_i A_k - \partial_k A_i), \quad (2.5)$$

the following components of the curvature tensor:

$$R_{juw}^i = \alpha B_{j\alpha}^i u^{\alpha-1}, \quad R_{jab}^i = 0, \quad \text{если } \{a\} \cup \{b\} \neq \{u, w\}, \quad (2.6)$$

and the following components of the curvature tensor at the point 0:

$$R_{iiu}^1(0) = 1, \quad R_{iiw}^2(0) = 1. \quad (2.7)$$

The computation of these values are direct. Note that there exists the following recurrent formula:

$$\nabla_{a_\alpha} \cdots \nabla_{a_1} R_{cab}^d = \partial_{a_\alpha} \nabla_{a_{\alpha-1}} \cdots \nabla_{a_1} R_{cab}^d + [\Gamma_{a_\alpha}, \nabla_{\partial_{a_{\alpha-1}}} \cdots \nabla_{\partial_{a_1}} R(\partial_a, \partial_b)]_c^d, \quad (2.8)$$

where Γ_{a_α} denote the operator with the matrix $(\Gamma_{ba_\alpha}^a)$.

Proof of the inclusion $\mathfrak{hol} \subset \mathfrak{g}^{\mathfrak{h}}$. The equality $\Gamma_{1b}^a = \Gamma_{2b}^a = 0$ means that the vector fields ∂_v, ∂_z are parallel, i.e. $\nabla \partial_v = \nabla \partial_z = 0$. According to the holonomy principle, \mathfrak{hol} annihilates the vectors $p_1 = (\partial_v)_0, p_2 = (\partial_z)_0$, this implies the inclusion $\mathfrak{hol} \subset \mathfrak{g}^{\mathfrak{so}(r,s)}$. It remains to prove that $\text{pr}_{\mathfrak{so}(r,s)} \mathfrak{hol} \subset \mathfrak{h}$, i.e.

$$\text{pr}_{\mathfrak{so}(r,s)} (\nabla_{\partial_{a_\alpha}} \cdots \nabla_{\partial_{a_1}} R(\partial_a, \partial_b)(0)) \in \mathfrak{h}$$

for all α . The equalities (2.2) and (2.3) show that

$$\begin{aligned} \nabla_{a_\alpha} \cdots \nabla_{a_1} R_{jab}^i &= \partial_{a_\alpha} \nabla_{a_{\alpha-1}} \cdots \nabla_{a_1} R_{jab}^i \\ &+ [\text{pr}_{\mathfrak{so}(r,s)} \Gamma_{a_\alpha}, \text{pr}_{\mathfrak{so}(r,s)} (\nabla_{\partial_{a_{\alpha-1}}} \cdots \nabla_{\partial_{a_1}} R(\partial_a, \partial_b))]_j^i. \end{aligned}$$

Note that if $\text{pr}_{\mathfrak{so}(r,s)} \Gamma_{a_\alpha} \neq 0$, then $a_\alpha = w$; in this case

$$\text{pr}_{\mathfrak{so}(r,s)} \Gamma_w = \sum_{\alpha=1}^N B_\alpha u^\alpha.$$

Now it is easy to prove the inclusion $\mathfrak{hol} \subset \mathfrak{g}^{\mathfrak{h}}$ using the induction and (2.3), (2.6).

Proof of the inclusion $\mathfrak{g}^{\mathfrak{h}} \subset \mathfrak{hol}$. Equalities (2.7) imply the inclusion $\{(0, X, Y, 0) | X, Y \in \mathbb{R}^{r,s}\} \subset \mathfrak{hol}$. Using (2.2), we get $\{(0, 0, 0, c) | c \in \mathbb{R}\} \subset \mathfrak{hol}$. From (2.3) and (2.8) follows the equality

$$(\nabla_u)^\beta R_{juw}^i = (\partial_u)^\beta R_{juw}^i = \sum_{\alpha=\beta+1}^N \alpha(\alpha-1) \cdots (\alpha-\beta) B_{j\alpha}^i u^{\alpha-\beta-1}$$

for all $0 \leq \beta \leq N - 1$. In particular,

$$(\nabla_u)^\beta R_{j_{uw}}^i(0) = (\beta + 1)! B_{j\beta+1}^i, \quad 0 \leq \beta \leq N - 1,$$

i.e.

$$\text{pr}_{\mathfrak{so}(r,s)}(\nabla_{\partial_u})^{\beta-1} R(\partial_u, \partial_w)(0) = \beta! B_\beta, \quad 1 \leq \beta \leq N.$$

Since the Lie algebra \mathfrak{h} is generated by the elements B_1, \dots, B_N , the last equality implies the inclusion $\mathfrak{g}^h \subset \mathfrak{hol}$. The theorem is true.

3. Correlation with the case of Lorentzian manifolds. Let us compare the obtained result with the classification of the holonomy algebras of Lorentzian manifolds [4, 10]. The holonomy algebra of an $(n + 2)$ -dimensional Lorentzian manifold is a subalgebra of the Lorentzian Lie algebra $\mathfrak{so}(1, n + 1)$, $n \geq 0$. Consider a basis p, e_1, \dots, e_n, q of the Minkowski space $\mathbb{R}^{1,n+1}$ such that the Minkowski metric η has only the following non-zero values: $\eta(p, q) = \eta(e_i, e_i) = 1$. The subalgebra $\mathfrak{so}(1, n + 1)_{\mathbb{R}p} \subset \mathfrak{so}(1, n + 1)$ preserving the isotropic line $\mathbb{R}p$ has the form

$$\left\{ \left(\begin{array}{ccc} a & -X^t & 0 \\ 0 & A & X \\ 0 & 0 & -a \end{array} \right) \middle| a \in \mathbb{R}, A \in \mathfrak{so}(n), X \in \mathbb{R}^n \right\} = (\mathbb{R} \oplus \mathfrak{so}(n)) \times \mathbb{R}^n.$$

It is enough to consider the subalgebras $\mathfrak{g} \subset \mathfrak{so}(1, n + 1)$ contained in $\mathfrak{so}(1, n + 1)_{\mathbb{R}p}$. For an arbitrary subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ consider the subalgebra

$$\mathfrak{g}^h = \left\{ \left(\begin{array}{ccc} 0 & -X^t & 0 \\ 0 & A & X \\ 0 & 0 & 0 \end{array} \right) \middle| A \in \mathfrak{h}, X \in \mathbb{R}^n \right\} = \mathfrak{h} \times \mathbb{R}^n \subset \mathfrak{so}(1, n + 1)_{\mathbb{R}p}.$$

Leistner [10] proved the following non-trivial statement: *if \mathfrak{g}^h is the holonomy algebra of a Lorentzian manifold, then the subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ must be the holonomy algebra of a Riemannian manifold.* The proof is based on the fact that the holonomy algebra $\mathfrak{g} \subset \mathfrak{so}(r, s)$ of an arbitrary pseudo-Riemannian manifold of signature (r, s) is generated by the images of algebraic curvature tensors, these tensors belong to the space $\mathcal{R}(\mathfrak{g})$ consisting of the 2-forms on $\mathbb{R}^{r,s}$ with the values in \mathfrak{g} and satisfying the first Bianchi identity

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0, \quad X, Y, Z \in \mathbb{R}^{r,s}.$$

For $R \in \mathcal{R}(\mathfrak{g}^h)$, the projection $\text{pr}_{\mathfrak{h}} \circ R$ is of the form

$$\text{pr}_{\mathfrak{h}} \circ R(p, \cdot) = 0, \quad \text{pr}_{\mathfrak{h}} \circ R|_{\mathbb{R}^n \times \mathbb{R}^n} \in \mathcal{R}(\mathfrak{h}), \quad \text{pr}_{\mathfrak{h}} \circ R(X, q) = P(X), \quad X \in \mathbb{R}^n,$$

where $P : \mathbb{R}^n \rightarrow \mathfrak{h}$ is a linear map satisfying the identity

$$\eta(P(X)Y, Z) + \eta(P(Y)Z, X) + \eta(P(Z)X, Y) = 0, \quad X, Y, Z \in \mathbb{R}^n.$$

Let $\mathcal{P}(\mathfrak{h})$ be the space of such maps P . We get that \mathfrak{h} must be generated by the images of the elements of the spaces $\mathcal{R}(\mathfrak{h})$ and $\mathcal{P}(\mathfrak{h})$. Leistner showed that from this condition it follows that \mathfrak{h} must be generated by the elements of the space $\mathcal{R}(\mathfrak{h})$; this means that $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a Riemannian manifold.

Consider now the subalgebra $\mathfrak{g}^{\mathfrak{h}} \subset \mathfrak{so}(2, n+2)$ from the previous section, where $\mathfrak{h} \subset \mathfrak{so}(n)$. It is easy to show that for $R \in \mathcal{R}(\mathfrak{g}^{\mathfrak{h}})$, the projection $\text{pr}_{\mathfrak{h}} \circ R$ satisfies

$$\text{pr}_{\mathfrak{h}} \circ R(p_1, \cdot) = \text{pr}_{\mathfrak{h}} \circ R(p_1, \cdot) = 0, \quad \text{pr}_{\mathfrak{h}} \circ R|_{\mathbb{R}^n \times \mathbb{R}^n} \in \mathcal{R}(\mathfrak{h}),$$

$$\text{pr}_{\mathfrak{h}} \circ R(\cdot|_{\mathbb{R}^n}, q_1), \text{pr}_{\mathfrak{h}} \circ R(\cdot|_{\mathbb{R}^n}, q_2) \in \mathcal{P}(\mathfrak{h}), \quad \text{pr}_{\mathfrak{h}} \circ R(q_1, q_2) = B \in \mathfrak{h}.$$

At the same time, the element $B \in \mathfrak{h}$ can be chosen in an arbitrary way; in order to see that it is enough to consider the following tensor $R \in \mathcal{R}(\mathfrak{g}^{\mathfrak{h}})$:

$$R(q_1, q_2) = (B, 0, 0, 0), \quad R(X, Y) = (0, 0, 0, 2\eta(BX, Y)), \quad R(p_1, \cdot) = R(p_2, \cdot) = 0,$$

$$R(X, q_1) = (0, 0, BX, 0), \quad R(X, q_2) = (0, -BX, 0, 0), \quad X, Y \in \mathbb{R}^n.$$

The belonging $R \in \mathcal{R}(\mathfrak{g}^{\mathfrak{h}})$ can be checked directly. For any pseudo-Riemannian manifold (M, g) with the holonomy algebra $\mathfrak{hol} \subset \mathfrak{so}(T_x M, g_x)$ at a point $x \in M$ we have

$$\nabla_{Z_\alpha} \cdots \nabla_{Z_1} R_x \in \mathcal{R}(\mathfrak{hol}), \quad \alpha \geq 0, \quad Z_1, \dots, Z_\alpha \in T_x M.$$

In the construction of the metric from the last section we used this property as well as the just described algebraic curvature tensors.

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