ON SMOOTH HYPERSURFACES OF INFINITE TYPE

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1. Introduction

In this paper, we study points of infinite type on certain real hypersurfaces in $\mathbb{C}^2$ at which those hypersurface possess a germ of a nontrivial symmetry. Our interest in those was sparked by a recent paper of Kang-Tae Kim and Ninh Van Thu, in which they proved that under certain assumptions, either such a real hypersurface is actually not only of infinite type, but nonminimal (i.e. contains a complex curve), or the symmetry is a rotation. For a discussion on how theis result is related to the Greene-Krantz conjecture, we would like to refer the reader to the above paper. However, their result assumed that a curvature condition holds at the infinite type point (it is, to be exact, a lineally convex point), and also a certain condition on the vanishing of certain terms in the Taylor series of its defining function. We therefore wanted to shed some light on those conditions in terms of our recent classification of so-called "ruled" hypersurfaces. In particular, we were able to show that neither of these conditions are actually necessary in the class of 1-nonminimal hypersurfaces.

This means that we obtain a geometric statement in the class of 1-nonminimal hypersurfaces, which we can explain as follows: Let $M \subset \mathbb{C}^2$ be a smooth hypersurface, $0 \in M$ a point of infinite type. This means that the hypersurface M contains a formal complex curve through 0, by a result of Fornaess. Our main result is that such a hypersurface which admits in addition a nontrivial infinitesimal symmetry (where nontrivial excludes certain trivial cases, namely rotations and translations) and whose associated formal hypersurface $\tilde{M}$ is 1-nonminimal actually contains a germ of a complex curve through 0. Before we state the theorem, let us recall what our assumption actually means.

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The defining equation of M close by 0 is given in local holomorphic coordinates (z, w) as

$$\operatorname{Im} w = P(z, \bar{z}) + (\operatorname{Re} w)A(z, \bar{z}) + O((\operatorname{Re} w)^2),$$

where P and A are germs of smooth, real-valued functions at $0 \in \mathbb{C}^2$. If M is of infinite type at 0, then $P(z, \bar{z})$ vanishes to infinite order at 0. M being formally 1-nonminimal means that the formal hypersurface defined by

$$\operatorname{Im} w = (\operatorname{Re} w)\tilde{A}(z, \bar{z}),$$

where $\tilde{A}(z, \bar{z}) \in Fz, \bar{z}$ is the Taylor series of the smooth function A, is 1-nonminimal (i.e. not Levi-flat).

Theorem 1.1. *Assume that $M \subset \mathbb{C}^2$ is a smooth hypersurface through $0 \in \mathbb{C}^2$ and that M is of infinite type at 0 and formally 1-nonminimal at 0. If there exists a nontrivial germ of a holomorphic vector field X at 0 whose real part is tangent to M near 0, then either there exist holomorphic coordinates in which X takes one of the forms*

$$X = iz\frac{\partial}{\partial z}, \quad X = \frac{\partial}{\partial z}$$

or M contains the germ of a holomorphic curve in $\mathbb{C}^2$ through 0.

Let us point out that this means in particular that if M is a germ of a smooth hypersurface through 0 which is of infinite type at 0 and which has a nontrivial holomorphic symmetry, then in suitable coordinates the defining equation of M takes one of the forms

$$\operatorname{Im} w = \varphi(z+\bar{z}, \operatorname{Re} w), \quad \operatorname{Im} w = \varphi(|z|^2, \operatorname{Re} w), \quad \operatorname{Im} w = \operatorname{Re} w\varphi(z, \bar{z}, \operatorname{Re} w).$$

If

$$X = \left(\sum_j \alpha_j(z)w^j\right)\frac{\partial}{\partial z} + \left(\sum_j \beta_j(z)w^{j+1}\right)\frac{\partial}{\partial w} = \sum_{j\geq m} X_j$$

is an infinitesimal symmetry of M split according to the weighted degree where z has weight 0 and w has weight 1, it necessarily is an infinitesimal symmetry of $\tilde{M}$, so the lowest degree homogeneous term X_m is tangent to $\operatorname{Im} w = \operatorname{Re} w\tilde{A}(z, \bar{z})$. These fields as well as the possible choices for $\tilde{A}$ have been studied and characterized completely in the article of the authors, so we have to show that if an X with such a lowest order term is tangent to M, then necessarily $P(z, \bar{z}) = 0$.

We are first going to prove some Lemmata which will settle that case $m \geq 1$. This splits naturally into two cases according to whether $\alpha_m(0) = 0$ or not. Since the latter one is simpler, we start with this case.

Lemma 1.2. *If $\alpha_m(0) \neq 0$, then $P(z,\bar z) = 0$ for all z close to* 0.

Proof. By the remark following Proposition 14 of [12], we can assume that we have chosen local holomorphic coordinates such that $X_m = w^m \frac{\partial}{\partial z}$. We now apply $X = X_m + \hat X$, where

$$\hat X = w^{m+1} r(z,w) \frac{\partial}{\partial z} + w^{m+2} s(z,w) \frac{\partial}{\partial w},$$

to the defining equation of M, we obtain the equations

$$(-1)^m (1 + iPr) P_z + (1 - iPr) P_{\bar z} = AP$$

$$(-1)^m \frac{-m + i(m+1)Pr + P^2 r_w}{Q + i} P_z + \frac{-m - i(m+1)P\bar r + P^2 \bar r_{\bar w}}{Q - i} P_{\bar z} = BP,$$

where A and B are smooth. Since the coefficient functions of P_z and $P_{\bar z}$ of these two equations are linearly independent at 0, we see that

$$P_z = CP, \quad P_{\bar z} = DP,$$

where C and D are smooth. By uniqueness of solutions to ODE, we conclude that $P(z,\bar z) = 0$. □

Lemma 1.3. *If $\alpha_m(0) = 0$, then $P(z,\bar z) = 0$ for all z close to* 0.

Proof. We can again use [12]. This time, it follows from the fact that X_m is tangent to $\operatorname{Im} w = \operatorname{Re} w\tilde A(z,\bar z)$ that $\tilde A(z,\bar z)$ is actually convergent and that one can choose holomorphic coordinates in which

$$X_m = \alpha_0 z w^m \frac{\partial}{\partial z} + w^{m+1} \frac{\partial}{\partial w}.$$

As before, we write $X = X_m + \hat X$, where

$$\hat X = r(z,w) w^{m+1} \frac{\partial}{\partial z} + s(z,w) w^{m+2} \frac{\partial}{\partial w},$$

and consider the equation $(\operatorname{Re} X)(v - P(z,\bar z) - uQ(z,\bar z,u) = 0$ if $v = P(z,\bar z) + uQ(z,\bar z,u)$. The terms constant and linear u in that equation yield:

$$(-1)^m (irP + \alpha z) P_z + (-i\bar r P + \bar\alpha \bar z) P_{\bar z} = AP$$

$$(-1)^m \frac{P^2 r_w - I(m+1)Pr - m\alpha z}{Q + i} P_z + \frac{P^2 \bar r_{\bar w} + I(m+1)Pr - m\bar\alpha z}{Q - i} P_{\bar z} = BP,$$

where $A(z,\bar z)$ and $B(z,\bar z)$ are smooth. Since P vanishes to infinite order at $z = 0$, we can rewrite this equation as

$$(-1)^m a(z) z P_z + \overline{a(z)} \bar z P_{\bar z} = AP$$

$$(-1)^m b(z) z P_z + \overline{b(z)} \bar z P_{\bar z} = BP,$$

where $a(0) = \alpha$ and $b(0) = -im\alpha$. Since

$$\begin{vmatrix} a(0) & \overline{a(0)} \\ b(0) & \overline{b(0)} \end{vmatrix} = \begin{vmatrix} \alpha & \overline{\alpha} \\ -im\alpha & im\bar{\alpha} \end{vmatrix} = 2im|\alpha|^2 \neq 0$$

we can write

$$zP_z = CP, \quad \bar{z}P_{\bar{z}} = DP,$$

for some smooth functions C and D. We claim that this implies that $P = 0$.

Evaluating along a line through z_0 and writing $\varphi(t) = P(tz_0)$, we see that φ satisfies the equation

$$t\varphi'(t) = \sigma(t)\varphi(t)$$

for a smooth function σ defined in a neighbourhood of $0 \in \mathbb{R}$. We claim that this implies $\varphi(t) = 0$ for small t. Indeed, if $\sigma(0) = 0$, $\varphi(t)$ satisfies the ODE $\varphi'(t) = \varphi(t)\hat{\sigma}(t)$ and since $\hat{\sigma}(t)$ is smooth and φ is flat at 0 it follows that $\varphi(t) = 0$. If on the other hand, $\sigma(0) = a \neq 0$ then if $\varphi(t) \neq 0$ for small positive (or negative) t, then by writing

$$\frac{\varphi'(t)}{\varphi(t)} = \frac{\sigma(t)}{t},$$

we see that $|\varphi(t)| \simeq |t|^a$, hence also in that case $\varphi = 0$. □

For the case $m = 0$, several additional Lemmata are needed.

Lemma 1.4. *Assume that $X = \alpha z\frac{\partial}{\partial z} + tw\frac{\partial}{\partial w} + \hat{X}$ with $\alpha \neq 0$ is tangent to $v = P + uQ$. Then after rescaling, $\alpha = i$, necessarily $P(z,\bar{z}) = P(|z|)$, and if $P \neq 0$, then $t = 0$.*

Proof. Since $\alpha z\frac{\partial}{\partial z} + tw\frac{\partial}{\partial w}$ is tangent to $v = uQ(z,\bar{z},0)$, we have that α is imaginary and t is real, and by rescaling, we can assume $\alpha = i$. Computation shows that there exist smooth functions φ and ψ such that

$$\operatorname{Re} iz\varphi P_z = \psi P,$$

where $\psi(0) = t$. Changing to polar coordinates (r,θ), we obtain $P_\theta = \tilde{\psi}P$ with $\tilde{\psi}(0) = t$. So necessarily $t = 0$. □

Lemma 1.5. *Assume that $X = iz\frac{\partial}{\partial z} + X_m + \hat{X}$ is given and that X_m is holomorphic. Then there exists a holomorphic change of coordinates in which X_m is given by*

$$X_m = \alpha z w^m \frac{\partial}{\partial z} + \beta w^{m+1}\frac{\partial}{\partial w}.$$

Proof. Let us introduce new coordinates by $(x, y) \in \mathbb{C}^2$ by

$$z = x + y^m f(x)$$
$$w = y + y^{m+1} g(x).$$

We compute the transformation formula for the basis vector fields up to terms homogeneous of degree m (where we assign weight 1 to y and weight 0 to x):

$$\frac{\partial}{\partial z} = (1 - y^m f'(x))\frac{\partial}{\partial x} - y^{m+1} g'(x)\frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial w} = -my^{m-1} f(x)\frac{\partial}{\partial x} + (1 - (m+1)y^m g(x))\frac{\partial}{\partial y}.$$

Hence, if $X_m = \alpha(z) w^m \frac{\partial}{\partial z} + \beta(z) w^{m+1}$, then in the new coordinates, X_m is given by

$$X_m = (\beta(x) - \beta(0) - xg'(x))\, y^m \frac{\partial}{\partial x} + (\alpha(x) - \alpha'(0)x - i(f(x) - xf'(x)))\, y^{m+1}\frac{\partial}{\partial y}.$$

Hence, we can obtain the desired form for suitable holomorphic $f(x)$ and $g(x)$. □

Lemma 1.6. *Assume that* $X = iz\frac{\partial}{\partial z} + X_m + \hat{X}$, *where*

$$X_m = \alpha z w^m \frac{\partial}{\partial z} + \beta w^{m+1}\frac{\partial}{\partial w},$$

has $\operatorname{Re} X$ *tangent to* $v = u\tilde{Q}(z, \bar{z}, u)$. *Then* $\operatorname{Re} X_m$ *is tangent to* $v = u\tilde{Q}(z, \bar{z}, 0)$ *and* $\tilde{Q}(z, \bar{z}, 0) = \tilde{Q}_1(|z|^2)$.

Proof. Let us write $\tilde{Q}(z, \bar{z}, u) = \sum_j \tilde{Q}_{j+1}(z, \bar{z}) u^j$. First, we consider the terms of homogeneity $k < m + 1$ in the tangency equation

$$(\operatorname{Re} X)(v - u\tilde{Q}(z, \bar{z}, u))\Big|_{v = u\tilde{Q}} = 0,$$

which are given by the terms of homogeneity k in

$$(\operatorname{Re} X_0)(v - u\tilde{Q}(z, \bar{z}, u))\Big|_{v = u\tilde{Q}} = 0.$$

Computation shows that this implies

$$\operatorname{Re} iz\tilde{Q}_{k,z} = 0.$$

Hence our claim of $\tilde{Q}_1(z, \bar{z}) = \tilde{Q}_1(|z|^2)$ is established.

Now the terms of homoegeneity $m+1$ in the tangency equation are simply

$$-u^{m+1}\operatorname{Re} iz\tilde{Q}_{m+1,z} + \operatorname{Re} X_m(v - u\tilde{Q}_1)\Big|_{v=u\tilde{Q}_1} = 0.$$

From the form of X_m and the fact that $Q_1(z,\bar{z}) = Q_1(|z|^2)$ we see that the second term cannot contain any monomials except for $u^{m+1}|z|^{2a}$. These are, however, zero in the first term. Hence $\operatorname{Re} X_m$ is tangent to $v = u\tilde{Q}_1$. Furthermore, necessarily $\tilde{Q}_{m+1}(z,\bar{z}) = \tilde{Q}_{m+1}(|z|^2)$. □

Lemma 1.7. *Assume that $X = iz\frac{\partial}{\partial z} + \alpha z w^m \frac{\partial}{\partial z} + \beta w^{m+1}\frac{\partial}{\partial w} + \hat{X}$ with $\operatorname{Re}\alpha \neq 0$ and $\beta \in \mathbb{R}\backslash\{0\}$ has $\operatorname{Re} X$ tangent to $v = P(|z|^2) + uQ(z,\bar{z},u)$. Then $P = 0$.*

Proof. We consider the terms homogeneous in degree 0 and 1 in the equation

$$(\operatorname{Re} X)(v - P - uQ)\Big|_{v=P+uQ} = 0.$$

These give rise to equations (where we substitute $x = |z|^2$)

$$\begin{aligned} xa(x,P)P' &= Pb(x,P) \\ xc(x,P)P' &= Pd(x,P). \end{aligned}$$

where

$$\begin{aligned} a(0,0) &= \alpha + (-1)^m\bar{\alpha} \\ b(0,0) &= \frac{i}{2}((-1)^m - 1) \\ c(0,0) &= \alpha + (-1)^{m+1}\bar{\alpha} \\ d(0,0) &= \frac{(m+1)}{2}((-1)^m - 1). \end{aligned}$$

Corresponding to whether m is even or odd, either the first or the second equation gives rise to a Fuchsian equation for P which has only the zero solution. □

We can now prove 1.1. If our vector field X is of the form $X = X_m + \hat{X}$ with the lowest order homogeneous term X_m being of degree $m \geq 1$, then according to whether $(w^{-m}X_m)(0) = 0$ or $(w^{-m}X_m)(0) \neq 0$, we apply 1.3 or 1.2, respectively, which shows that in either case $P = 0$.

We are thus left with the case that $X = X_0 + \hat{X}$. If $X_0(0) \neq 0$, we can choose holomorphic coordinates in which $X = \frac{\partial}{\partial z}$. If on the other hand, $X_0(0) = 0$, we know from [12] that in suitable holomorphic coordinates, $X_0 = iz\frac{\partial}{\partial z} + tw\frac{\partial}{\partial w}$. Now we apply 1.4 in order to see that if $P \neq 0$, then necessarily $t = 0$. Hence we only need to deal with that case. By applying 1.5 inductively, we see that either there exist holomorphic coordinates in which $X = iz\frac{\partial}{\partial z} + X_m + \hat{X}$ with $X_m \neq 0$ or formal coordinates for which $X = iz\frac{\partial}{\partial z}$. In the later case, there also exist holomorphic coordinates with that property.

If on the other hand $X_m \neq 0$, we have

$$X_m = \alpha z w^m \frac{\partial}{\partial z} + \beta w^{m+1} \frac{\partial}{\partial w}.$$

By 1.6, it follows that X_m is tangent to the formal hypersurface $\operatorname{Im} w = \operatorname{Re} w Q_1(|z|^2)$. It follows now from [12] that necessarily $\beta \in \mathbb{R} \setminus \{0\}$ and $\operatorname{Re} \alpha \neq 0$. Hence we are in the position to apply 1.7 in order to conclude that in this case $P = 0$.

References

[1] Baouendi, M. S., Ebenfelt, P., Rothschild, L. P., *Convergence and finite determination of formal CR mappings*, J. Amer. Math. Soc. 13, (2000), 697-723

[2] Baouendi, M. S., Ebenfelt, P., Rothschild, L. P., *Local geometric properties of real submanifolds in complex space*, Bull. Amer. Math. Soc. (N.S.) **37** 3 (2000), 309–336.

[3] Bloom, T. and Graham, I., *On "type" conditions for generic real submanifolds of C^n*, Invent. Math. 40 (1977), no. 3, 217–243.

[4] E.Cartan : *Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes, I* , Ann. Math. Pura Appl. **11** (1932), p. 17–90.

[5] Catlin, D., *Boundary invariants of pseudoconvex domains*, Ann. Math. **120** (1984), 529–586.

[6] Chern, S. S. and Moser, J., *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271.

[7] D'Angelo, J., *Orders od contact, real hypersurfaces and applications*, Ann. Math. **115** (1982), 615–637.

[8] Kim, K.-T.,

[9] Kohn, J. J., *Boundary behaviour of $\bar{\partial}$ on weakly pseudoconvex manifolds of dimension two*, J. Differential Geom. **6** (1972), 523–542.

[10] S.Y.Kim, D.Zaitsev, *Equivalence and embedding problems for CR-structures of any codimension*, Topology **44** (2005), p. 557-584.

[11] Kolář, M. *The Catlin multitype and biholomorphic equivalence of models,*

[12] Kolář, M., Lamel, B., *Holomorphic equivalence and nonlinear symmetries of ruled hypersurfaces in $\mathbb{C}^2$* J. Geom. Anal., to appear.

[13] Kolář, M., *Normal forms for hypersurfaces of finite type in $\mathbb{C}^2$*, Math. Res. Lett. **12** (2005), p. 523-542

[14] Kolář, M., Meylan, F., Zaitsev, D., *Chern-Moser operators and polynomial models in CR geometry*, submitted.

[15] H.Poincaré : *Les fonctions analytique de deux variables et la représentation conforme* Rend. Circ. Mat. Palermo **23** (1907), p. 185-220

[16] A.G.Vitushkin : *Real analytic hypersurfaces in complex manifolds*, Russ. Math. Surv. **40** (1985), p. 1-35

[17] S.M.Webster : *On the Moser normal form at a non-umbilic point*, Math.Ann **233** (1978), p. 97-102

[18] Wells, R. O., Jr., *The Cauchy-Riemann equations and differential geometry*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), 187–199.

[19] P.Wong : *A construction of normal forms for weakly pseudoconvex CR manifolds in* $\mathbb{C}^2$, Invent. Math. **69** (1982), p. 311-329.

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