AUTOMORPHISM GROUPS OF LEVI DEGENERATE HYPERSURFACES IN $\mathbb{C}^3$

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Abstract. We give a complete description of hypersurfaces of finite Catlin multitype, which admit nonlinear nonrigid symmetries. As a consequence, we obtain a complete classification of possible symmetry groups of such manifolds.

1. Introduction

By the classical results of Chern and Moser, Beloshapka and Kruzhilin, symmetries of strongly pseudoconvex nonspherical hypersurfaces are linear in normal coordinates, as defined by Chern and Moser. Moreover, the symmetries of the sphere itself are generated by holomorphic vector fields with at most quadratic coefficients. Hence, on an arbitrary strongly pseudoconvex hypersurface, automorphisms are determined by their 2-jets.

More recently, automorphism groups have been studied on Levi degenerate manifolds of finite type. In complex dimension two, the situation is quite similar to the nondegenerate case. Except for the sphere and its higher type analogs given by $Im\ w = |z|^k$, all automorphisms are linear in normal coordinates, constructed in [24].

In higher dimensions Levi degenerate manifolds in general admit nonlinear symmetries generated by vector fields with coefficients of arbitrarily high degree.

On the other hand, the results of [22] give a sharp jet determination result, which shows that in general symmetries are determined by weighted jets of order at most one, which gives some hope to classify all nonlinear symmetries explicitely.

Our aim in this paper is to study and classify nonlinear symmetries of Levi degenerate hypersurfaces in complex dimension three.

Nonlinear vector fields which are real tangent to M fall naturally into two classes. If its coefficients depend only on complex tangential variables, the vector field is called rigid. This class of vector fields is analysed in [23]. In this paper we study nonlinear vector fields which are not rigid. We obtain a complete classification of hypersurfaces of finite multitype which admit such symmetries.

Combining the results of this paper with [22] and [23] gives a complete understanding of possible symmetry groups for hypersurfaces of finite multitype in $\mathbb{C}^3$.

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2. Nontransversal shifts

We refer the reader to [22] for the notation used below.

In this section we describe all hypersurfaces in $\mathbb{C}^3$ for which a nontransversal shift can be integrated.

Lemma 2.1. *Let $X = i\partial_{z_1}$ be a nontransversal vector field such that $ReX(P) = 0$. If there is a vector field Y in $aut(M_H, p)$ such that $[Y, W] = X$, then P has the form*

$$P(z, \bar{z}) = P_0(z_2, \bar{z}_2) + x_1 P_1(z_2, \bar{z}_2) + x_1^2 P_2(z_2, \bar{z}_2). \tag{2.1}$$

Proof. By assumption, we may write P as

$$P(z, \bar{z}) = \sum_{j=0}^{m} x_1^j P_j(z', \bar{z}'), \tag{2.2}$$

where $P_m \neq 0$. Suppose, by contradiction, that $m > 2$. We split Y according to the powers of z_1, writing

$$Y = iw\partial_{z_1} + \sum_{j=-m}^{k} Y_j, \tag{2.3}$$

where Y_j is of the form

$$Y_j = {\varphi_1}^j(z_2){z_1}^{j+1}\partial_{z_1} + {\varphi_2}^j(z_2){z_1}^j\partial_{z_2} + \psi^j(z_2){z_1}^{j+m}\partial_w, \tag{2.4}$$

with ${\varphi_1}^j(z_2) = {\varphi_1}^j(z_2) = 0$ for $j < 0$. We notice that $2m - 1 \leq m + k$. Indeed, if not, applying $\mathsf{Re}\, Y$ to $P - v$ the first term of the right handside of (2.3) will contain a term $-mP_m^2 {x_1}^{2m-1}$ which can not cancel, since all other terms are of maximal power $m + k$ with respect to z_1.

We next claim that ${\varphi_1}^k(z_2) = 0$ and P_m is constant. Indeed, we have

$$\mathsf{Re}\, Y_k(x_1^m P_m) = 0, \tag{2.5}$$

since $\mathsf{Re}\, Y(P - v) = 0$. We observe that, since $m > 2$, $\mathsf{Re}\, \psi^k(z_2){z_1}^{k+m}$ cannot contain terms in $x_1 {y_1}^{m+k-1}$ and in ${y_1}^{m+k}$. Hence $\psi^k(z_2) = 0$.

Expanding 2.5 we obtain

$$\delta_{k,m-1} m x_1^{2m-1} P_m^2 = m x_1^{m-1} P_m \mathsf{Re}\, \varphi_1^k z_1^k + x_1^m \mathsf{Re}\, \varphi_2^k z_1 \frac{\partial P_m}{\partial z_2} \tag{2.6}$$

Comparing degrees in z^2, we see that $wt.\varphi_1^k = wt.P_m$. Looking at coefficients of $x^{m-1}y^{k+1}$ we obtain that φ_1^k is a constant, hence P_m is a constant.

Collecting terms of degree $2m - 1$ with respect to z_1, we obtain

$$\begin{aligned} -mP_m^2 {x_1}^{2m-1} = {x_1}^{m-1} &\mathsf{Re}\,(\alpha_{m-1}(z_2){z_1}^m) + \dots \\ &+ x_1 \mathsf{Re}\,(\alpha_1(z_2){z_1}^{2m-2}) + \mathsf{Re}\,(\alpha_0(z_2){z_1}^{2m-1}). \end{aligned} \tag{2.7}$$

Comparing weighted degrees on both sides we obtain that $\alpha_j(z_2),\ j=0,\dots,m-1$, are constant.

We claim that $\alpha_j(z_2),\ j=0,\dots,m-1$, are uniquely determined by (2.7).

$$\alpha_{m-2} = \mathsf{Re}\,\lambda(z_2)P_{m-1} - \mathsf{Re}\,\mu(z_2)\partial_{z_2}P_{m-2}, \tag{2.8}$$

where the right handside is homogeneous. Therefore, either P_{m-1} is constant or $\partial_{z_2}P_{m-2}$ is constant. Both cases lead to a contradiction, for P is a weighted homogeneous polynomial. This achieves the proof of the lemma. □

Lemma 2.2. *In the notation of the previous Lemma, we have $P_2 = constant$.*

Proof. By comparing the coefficients of degree $2+k$ with respect to z_1, with k given by (2.3), we obtain one of the following equation, depending of the value of k:

$$2x_1^3P_2^2 = 2x_1P_2\mathsf{Re}\,\varphi_1^2z_1^2 + x_1^2\mathsf{Re}\,\varphi_2^1z_1\frac{\partial P_2}{\partial z_2} + \mathsf{Re}\,(h(z_2)z_1^3), \tag{2.9}$$

if $k=1$, or

$$0 = x_1^2\mathsf{Re}\,(\varphi_2^kz_1^k\frac{\partial P_2}{\partial z_2}) + 2x_1P_2\mathsf{Re}\,(\varphi_1^{k+1}{z_1}^{k+1}) + \mathsf{Re}\,(h(z_2){z_1}^{k+2}) \tag{2.10}$$

if $k\neq 1$. Suppose that the first equation (2.9) holds. We notice that if $h(z_2)\neq 0$ in (2.9) then it is a constant, by comparing terms in y_1^3. Using the fact that Y has weight $1-\mu_1$, and the fact that $z_1^3\partial_w$ has weight $3\mu_1-1$, we obtain that $\mu_1 = \frac{1}{2}$, and hence P_2 is a constant.

Assume now that $h(z_2)=0$.

Again, comparing degrees in z^2, we see that $wt.\varphi_1^1 = wt.P_2$. Looking at coefficients of $x_1y_1^2$ we obtain that φ_1^1 is a constant, hence P_2 is a constant.

If the second equation (2.10) holds, then, by looking to the terms of the form y_1^{2+k}, we see that $h(z_2)$ is a constant. Comparing the term in $x_1y_1^{2+k-1}$, we obtain $P_2\mathsf{Re}\,(\varphi_1^{k+1})$ is a constant. Therefore P_2 is constant since $\mathsf{Re}\,(\varphi_1^{k+1})\neq 0$. This achieves the proof of the lemma. □

Lemma 2.3. *Let $X = i\partial_{z_1}$ be a nontransversal vector field such that* $\mathsf{Re}\,X(P)=0$. *Let P be of the form*

$$P(z,\bar z) = x_1P_1(z_2,\bar z_2) + P_0(z_2,\bar z_2) \tag{2.11}$$

There is a vector field Y in $aut(M_H,p)$ such that $[Y,W]=X$, if and only if $P_1(z')$ is harmonic and P_0 admits a complex reproducing field, hence

$$P = (\mathsf{Re}\,z_1)\mathsf{Re}\,h(z_2) + c|z_2|^{l+1} \tag{2.12}$$

Proof. We obtain by integrating X

$$Y = iw\partial_{z_1} + \sum_{j=1}^{2}\varphi_j\partial_{z_j} + \psi\partial_w \tag{2.13}$$

Applying Y to $P-v$ we obtain from $\mathsf{Re}\, Y(P-v)=0$, using $\mathsf{Re}\, X(P)=0$,

$$P_0P_1 + x_1P_1^2 = \mathsf{Re}\,\varphi_2\frac{\partial P_0}{\partial z_2} + \mathsf{Re}\,(\varphi_1^0 + z_1\varphi_1^1)P_1 + x_1\mathsf{Re}\,\varphi_2\frac{\partial P_1}{\partial z_2} + \frac{1}{2}Im\psi \tag{2.14}$$

Comparing the coefficients of the constant terms and coefficients of x_1, y_1 we obtain

$$P_0P_1 = \mathsf{Re}\,\varphi_2^0\frac{\partial P_0}{\partial z_2} + \mathsf{Re}\,\varphi_1^0P_1 + \frac{1}{2}Im\psi_1^0. \tag{2.15}$$

and

$$x_1P_1^2 = x_1\mathsf{Re}\,\varphi_2^0\frac{\partial P_1}{\partial z_2} + \mathsf{Re}\,(\varphi_1^1z_1)P_1 + \varphi_2^1z_1\frac{\partial P_0}{\partial z_2} + \frac{1}{2}Im\psi_1^1. \tag{2.16}$$

Let k be as in (2.3). First assume $k>1$. Then we get

$$0 = \mathsf{Re}\,(\varphi_1^kz_1^k)P_1 + x_1\mathsf{Re}\,\sum_{j=2}^{n}\varphi_j^{k-1}z_1^{k-1}\frac{\partial P_1}{\partial z_j} + \frac{1}{2}Im\psi_1^kz_1^k. \tag{2.17}$$

Looking at the coefficient of $\bar{z}_1z_1^{k-1}$, we see that the middle term is zero. Then, we see that $\mathsf{Re}\,(\varphi_1^kz_1^k)P_1$ is harmonic, which implies that P_1 is constant.

Now let $k=1$. We get

$$x_1P_1^2 = x_1\mathsf{Re}\,\varphi_2^0\frac{\partial P_1}{\partial z_j} + \mathsf{Re}\,(\varphi_1^1z_1)P_1 + \frac{1}{2}Imz_1\psi_1^1 \tag{2.18}$$

Comparing coefficients of y_1, we get

$$Im\varphi_1^1P_1 + \frac{1}{2}\mathsf{Re}\,\psi_1^1 = 0.$$

This implies that P_1 is harmonic, $P_1 = c\mathsf{Re}\,\varphi_1^1$.

Now let $k=0$. Then

$$x_1P_1^2 + P_0P_1 = x_1\mathsf{Re}\,\varphi_2^0\frac{\partial P_1}{\partial z_j} + \mathsf{Re}\,\varphi_2^0\frac{\partial P_0}{\partial z_j} + \mathsf{Re}\,(\varphi_1^0z_1)P_1 + \frac{1}{2}Im\psi_1^0 \tag{2.19}$$

which gives

$$P_1^2 = \mathsf{Re}\,\varphi_2^0\frac{\partial P_1}{\partial z_j} \tag{2.20}$$

which implies that $P_1 = 0$, since the left hand is positive, and the right hand side contains no diagonal terms, which is a contradiction.

We return to the case $k \geq 1$ and continue the proof. using that P_1 is harmonic. For terms of order zero we use equation (2.15),

$$P_0P_1 = \mathsf{Re}\,(\varphi_1^0)P_1 + \mathsf{Re}\,\varphi_2^0\frac{\partial P_0}{\partial z_j} + Im\psi^0 \tag{2.21}$$

We obtain, using the form of P_1,

$$P_0 Im(\alpha z_2^l) = \mathsf{Re}\,(\delta z_2^{l+1})P_1 + \mathsf{Re}\,(\beta z_2^{l+1}\frac{\partial P_0}{\partial z_2}) + Im\gamma z_2^{2l+1}. \tag{2.22}$$

Write P_0 as

$$P_0(z_2,\bar z_2) = \sum_{j=j_0}^{l+1} A_j z_2^j \bar z_2^{l+1-j} \tag{2.23}$$

and $A_{j_0} \neq 0$. Substituting into () and comparing coefficients of $z_2^{j_0}\bar z_2^{l+1}$ we obtain

$$\alpha = j_0\beta, \qquad \bar\alpha = (l+1-j_0)\bar\beta \tag{2.24}$$

which gives

$$2a = l+1 \tag{2.25}$$

which means that $P_0 = c|z|^{l+1}$. To conclude, we have proved that P has the form

$$P = (\mathsf{Re}\,z_1)\mathsf{Re}\,h(z_2) + c|z_2|^{l+1} \tag{2.26}$$

where h is holomorphic.

□

Lemma 2.4. *Let $X = i\partial_{z_1} + h\partial_w$ be a nontransversal vector field such that $\mathsf{Re}\,X(P) = 0$ given in multitype coordinates. Let P has the form*

$$P(z,\bar z) = |z_1|^2 + \mathsf{Re}\,z_1 P_1(z_2,\bar z_2) + P_0(z_2,\bar z_2) \tag{2.27}$$

Then there is a vector field Y in $aut(M_H,p)$ such that $[Y,W] = X$, if and only if $P_1 = 0$ and P_0 admits a complex reproducing field, hence P has the form

$$P(z,\bar z) = |z_1|^2 + \sum_{|\alpha|=|\beta|} A_{\alpha,\beta} z^\alpha \bar z^\beta. \tag{2.28}$$

Proof. We have

$$X = i\partial_{z_1} - 2z_1\partial_w. \tag{2.29}$$

Hence

$$Y = iw\partial_{z_1} - 2z_1 w\partial_w + \varphi_1\partial_{z_1} + \varphi_2\partial_{z_2} + \psi\partial_w \tag{2.30}$$

Applying $\mathsf{Re}\,Y$ to $P - v$ we obtain

$$(2x_1 + \frac{1}{2}P_1)(|z_1|^2 + \mathsf{Re}\,z_1P_1(z_2,\bar z_2) + P_0(z_2,\bar z_2)) + \mathsf{Re}\,\psi_1\frac{\partial P}{\partial z_1} + \mathsf{Re}\,\psi_2\frac{\partial P}{\partial z_2} = 0 \tag{2.31}$$

since $\varphi = 0$ because P contains no harmonic terms. Comparing terms of second order in z_1 we obtain

$$2x_1^2 P_1 + \frac{1}{2}P_1|z_1|^2 + \mathsf{Re}\,(\varphi_1^1 z_1)\bar z_1 + \mathsf{Re}\,\varphi_2^1 z_1 x_1 = 0. \tag{2.32}$$

for the coefficient of y_1^2 we obtain

$$2P_1 + \frac{1}{2}P_1 + \mathsf{Re}\,\varphi_1 = 0 \tag{2.33}$$

hence P_1 is harmonic, therefore $P_1 = 0$, since we are in multitype coordinates.

Further, from the coefficients of first order terms in z_1 we obtain

$$2x_1 P_0 + \mathsf{Re}\, z_1 \psi_2 \frac{\partial P_0}{\partial z_2} = 0 \tag{2.34}$$

which gives

$$P_0 = \psi_2 \frac{\partial P_0}{\partial z_2}, \tag{2.35}$$

hence P_0 has a complex reproducing field, as claimed. □

Combining the above results with [22] and [23], we obtain a complete classification of hypersurfaces with nonlinear symmetries, including information about the dimensions of the symmetry groups.

References

[1] Baouendi, M. S., Ebenfelt, P., Rothschild, L. P., *Local geometric properties of real submanifolds in complex space*, Bull. Amer. Math. Soc. (N.S.) **37** (2000), 309–336.

[2] Bedford, E., Pinchuk, S. I., *Convex domains with noncompact groups of automorphisms*, Mat. Sb., **185** (1994), 3–26.

[3] Beloshapka, V. K., Ezhov, V. V., Schmalz, G., *Holomorphic classification of four-dimensional surfaces in* $\mathbb{C}^3$, Izv. Ross. Akad. Nauk Ser. Mat., **72** (2008), 3–18.

[4] Beloshapka, V. K., Kossovskiy, I. G., *Classification of homogeneous CR-manifolds in dimension 4*, J. Math. Anal. Appl., **374** (2011), 655–672.

[5] Bloom, T., Graham, I., *On "type" conditions for generic real submanifolds of* C^n, Invent. Math. **40** (1977), 217–243.

[6] Beals, M., Fefferman, C., Graham R., *Strictly pseudoconvex domains in* $\mathbb{C}^n$, Bull. Amer. Math. Soc. (N.S.) **8** (1983), 125–322.

[7] Cartan, E., *Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes, I* , Ann. Math. Pura Appl. **11** (1932), 17–90.

[8] Catlin, D., *Boundary invariants of pseudoconvex domains*, Ann. Math. **120** (1984), 529–586.

[9] Catlin, D., *Subelliptic estimates for* $\bar{\partial}$*-Neumann problem on pseudoconvex domains* , Ann. Math. **126** (1987), 131–191.

[10] Chern, S. S. Moser, J., *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271.

[11] D'Angelo, J., *Orders of contact, real hypersurfaces and applications*, Ann. Math. **115** (1982), 615–637.

[12] Ebenfelt, P., *New invariant tensors in CR structures and normal form for real hypersurfaces at a generic Levi degeneracy*, J. Diff. Geom. **50** (1998), 207–247.

[13] Ebenfelt, P., Lamel, B., Zaitsev, D., *Degenerate real hypersurfaces in* $\mathbb{C}^2$ *with few automorphisms*, Trans. Amer. Math. Soc. **361** (2009), 3241–3267.

[14] Fefferman, C., *Parabolic invariant theory in complex analysis*, Adv. in Math. **31** (1979), 131–262.

[15] Fels, G., Kaup, W., *Classification of Levi degenerate homogeneous CR manifolds in dimension 5*, Acta Math. **201** (2008), 1–82.

[16] Huang, X., and Yin, W., *A Bishop surface with a vanishing Bishop invariant*, Invent. Math. **176** (2009), 461–520.

[17] Isaev, A. V., Krantz, S. G., *Domains with non-compact automorphism group: a survey*, Adv. Math., **146**, (1999), 1–38.

[18] Jacobowitz, H., *An Introduction to CR structures*, Mathematical Surveys and Monographs, 32. Amer. Math. Soc., Providence, RI, 1990.

[19] Kohn, J. J., *Boundary behaviour of $\bar{\partial}$ on weakly pseudoconvex manifolds of dimension two*, J. Differential Geom. **6** (1972), 523–542.

[20] Kohn, J. J., *Subellipticity of the $\bar{\partial}$-Neumann problem on pseudoconvex domains: sufficient conditions*, Acta Math. **142** (1979), 79–122.

[21] Kim, S.Y., Zaitsev, D., *Equivalence and embedding problems for CR-structures of any codimension*, Topology **44** (2005), 557–584.

[22] Kolář, M., Meylan, F., Zaitsev, D., *Chern-Moser operators and polynomial models in CR geometry*, submitted.

[23] Kolář, M., Meylan, F., *A classification of generalized rotations in $\mathbb{C}^3$*

[24] Kolář, M., *The Catlin multitype and biholomorphic equivalence of models*, Int. Math. Res. Not. IMRN **18** (2010), 3530–3548.

[25] Kolář, M., *Normal forms for hypersurfaces of finite type in $\mathbb{C}^2$*, Math. Res. Lett., **12** (2005), 523–542.

[26] Kolář, M., Meylan, F., *Chern-Moser operators and weighted jet determination problems*, Geometric analysis of several complex variables and related topics, 75–88, Contemp. Math. 550, 2011.

[27] Kruzhilin, N. G., Loboda, A. V., *Linearization of local automorphisms of pseudoconvex surfaces*, Dokl. Akad. Nauk SSSR, **271** (1983), 280–282.

[28] Nadel, A. M., *Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature*, Ann. of Math. (2) **132** (1990), no. 3, 549–596.

[29] Poincaré, H., *Les fonctions analytiques de deux variables et la représentation conforme,* Rend. Circ. Mat. Palermo **23** (1907), 185–220.

[30] Siu, Y.-T., *Invariance of plurigenera*, Invent. Math. **134** (1998), no. 3, 661–673.

[31] Stanton, N., *Infinitesimal CR automorphisms of real hypersurfaces*, Amer. J. Math. **118** (1996), 209–233.

[32] Treves, F., *A treasure trove of geometry and analysis: the hyperquadric*, Notices Amer. Math. Soc. **47** (2000), 1246–1256.

[33] Vitushkin, A.G., : *Real analytic hypersurfaces in complex manifolds*, Russ. Math. Surv. **40** (1985), 1–35.

[34] Webster, S.M., *On the Moser normal form at a non-umbilic point*, Math. Ann. **233** (1978), 97–102.

[35] Yang, P., *Automorphism of tube domains*, Amer. J. Math. **104** (1982), 1005–1024.

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