ON NONLINEAR SYMMETRIES OF POLYNOMIAL MODELS

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Abstract. We consider polynomial models in $\mathbb{C}^3$ of the form $Im\ w = Re\ P(z,\bar z)\overline{Q(z,\bar z)}$, where P and Q are weighted homogeneous polynomials of different degrees. We give a complete characterization in terms of P and Q of such manifolds admit nonlinear rigid symmetries. Then we prove corresponding characterization of hypersurfaces admitting a rigid symmetry of zero weight.

1. Introduction

The study of possible complexity of automorphisms of CR maniolds has a long history. The classical case of Levi nondegenerate hypersurfaces was studied by Poincaré, Cartan, Tanaka, Chern and Moser, Vitushkin and many others. Most results on symmetries in this class are negative, showing that interesting symmetries are very rare. In particular, Beloshapka and Kruzhilin showed that if the hypersurface is not locally spherical, than its symmetries are linear in Chern-Moser normal coordinates.

Similar results where obtained for finite type hypersurfaces in $\mathbb{C}^2$. In particular, all such hypersurfaces which admit a nonlinear symmetry are biholomophically equivalent of the model $Im\ w = |z|^k$.

Several recent results indicate that the situation is a lot more interesting for finite type hypersurfaces in higher dimensions, and also for infinite type hypersurfaces in $\mathbb{C}^2$ (see e.g. [12], [14]).

On the one hand, the simple example of a finite type hypersurface in $\mathbb{C}^3$

$$Im\ w = Re\ z_1 \bar z_2^l$$

which admits an infinitesimal automorphism of the form

$$Y = i z_2^l \partial_{z_1},$$

where l is an arbitrary integer, shows that infinitesimal automorphisms may have coefficients of arbitrary degree. On the other hand, as the results of [14] show, the weighted degree of such coefficients is controlled by the Catlin multitype of the manifold.

By the results of [14], the Lie algebra of infinitesimal symmetries may in general contain six types of components, compared to five in the classical case of the sphere. The sixth

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component is formed by nonlinear vector fields acting in the complex tangential variable. Such vector fields will be called generalized rotations.

Our aim in this paper is to study this phenomenon in detail in the simplest case of hypersurfaces generalizing in a natural way the above example. We consider hypersurfaces of finite multitype in $\mathbb{C}^3$, given by

$$Im\ w = Re\ P(z,\bar z)\overline{Q(z,\bar z)}, \tag{1.1}$$

where P and Q are weighted homogeneous polynomials.

We give an explicit description, in terms of P and Q, of all such hypersurfaces which admit generalized rotations. We also give conditions under which such manifolds admit a rotation, i.e. a rigid vector field of weight zero.

2. Formulation of the problem

Let us first recall some needed definitions from [14].

Let $M \subseteq \mathbb{C}^3$ be a real analytic hypersurface, and $p \in M$ be a point of finite type $m \geq 2$ in the sense of Kohn and Bloom-Graham ([1], [3], [9]).
We consider local holomorphic coordinates (z, w) vanishing at p, where $z = (z_1, z_2)$ and $z_j = x_j + iy_j$, $j = 1, 2$, and $w = u + iv$. The hyperplane $\{v = 0\}$ is assumed to be tangent to M at p, hence M is described near p as the graph of a uniquely determined real valued function

$$v = \varphi(z_1, z_2, \bar z_1, z_2, u),\ d\varphi(0) = 0. \tag{2.1}$$

We cay assume that (see e.g. [1])

$$\varphi(z_1, z_2, \bar z_1, \bar z_2, u) = P_m(z, \bar z) + o(u, |z|^m), \tag{2.2}$$

where $P_m(z, \bar z)$ is a nonzero homogeneous polynomial of degree m without pluriharmonic terms.

Definition 2.1. Let X be a holomorphic vector field of the form

$$X = \sum_{j=1}^{2} f^j(z, w)\partial_{z_j} + g(z, w)\partial_w. \tag{2.3}$$

We say that X is rigid if f^1, f^2, g are all independent of the variable w.

Note that the rigid vector field W, of homogeneous weight -1, given by

$$W = \partial_w \tag{2.4}$$

lies in $\mathrm{aut}(M_P, p)$. We will denote by E the weighted homogeneous vector field of weight 0 defined by

$$E = \sum_{j=1}^{n} \mu_j z_j \partial_{z_j} + w\partial_w. \tag{2.5}$$

E is the weighted Euler field. Note that by definition of μ_j, E is a non rigid vector field lying in $\mathrm{aut}(M_P, p)$.

We can divide homogeneous rigid vector fields into three types, and introduce the following terminology.

Definition 2.2. Let $X \in \mathrm{aut}(M_P, p)$ be a rigid weighted homogeneous vector field. X is called

(1) a *shift* if the weighted degree of X is less than zero;
(2) a *rotation* if the weighted degree of X is equal to zero;
(3) a *generalized rotation* if the weighted degree of X is bigger than zero and less than one.

Note that in the Levi nondegenerate case (where $P(z, \bar{z}) = \langle z, z \rangle$ is a quadratic form), generalized rotations do not occur.

We will consider the following problem: Characterize P and Q, holomorphic polynomials in $z \in \mathbb{C}^n,\ n \geq 3$, such that $\mathsf{Re}\, P\overline{Q}$ is a weighted homogeneous polynomial of weighted degree one and

$$M = \{(z, w) \in \mathbb{C}^{n+1} : \mathsf{Im}\, w = P\overline{Q} + Q\overline{P}\} \tag{2.6}$$

is a holomorphically non degenerate hypersurface that admits either rotations, or generalized rotations or non transversal shifts.

3. Rotations

Lemma 3.1. *Let M be given by* (2.6)*. If M is holomorphically non degenerate, then necessarily $n = 2$.*

Proof. If $n > 2$, we claim that there exists a non zero field X such that

$$X(P) = X(Q) = 0, \tag{3.1}$$

which will contradict the fact that M is holomorphically nondegenerate. Writing

$$X = \sum_{j=1}^{n} a_j(z) \partial_{z_j}, \tag{3.2}$$

$$X(P) = 0 \Longleftrightarrow \sum_{j=1}^{n} a_j(z) \partial_{z_j} P = 0 \Longleftrightarrow (a_j) \perp (\partial_{z_j} P), \tag{3.3}$$

$$X(Q) = 0 \Longleftrightarrow \sum_{j=1}^{n} a_j(z) \partial_{z_j} Q = 0 \Longleftrightarrow (a_j) \perp (\partial_{z_j} Q).$$

These two conditions are always satisfied if $n > 2$. This achieves the proof of the lemma. □

Theorem 3.2. *Let M be given by* (2.6)*, and let*

$$R = \sum_{j=1}^{2} \mu_j z_j \partial_{z_j} \tag{3.4}$$

be the Euler real reproducing field. Then

- *If R is a complex reproducing field, M admits a rotation.*
- *If R is not a complex reproducing field, M admits a rotation if and only if $\mathsf{Re}\, P\overline{Q}$ is a homogeneous polynomial with respect to z_1, $\overline{z_1}$.*

Proof. Let X be a rotation. Without loss of generality, we may assume that it is given in Jordan form, since there is no non linear rotations up to a change of multitype coordinates (see...). Applying X to $\mathsf{Im}\, w - P\overline{Q} - Q\overline{P}$, we obtain

$$X(P)\overline{Q} + X(Q)\overline{P} + \overline{X}(\overline{P})Q + \overline{X}(\overline{Q})P = 0. \tag{3.5}$$

Since $\mathsf{Re}\, P\overline{Q}$ is a weighted homogeneous polynomial of weighted degree one, P and Q are also weighted homogeneous polynomials of weighted degree one. We claim that if P and Q have same weighted degree, that is, if R is a complex reproducing field, then there is always rotations. Indeed any vector field of the form

$$iaz_1\partial_{z_1} + ia\frac{\mu_2}{\mu_1} z_2\partial_{z_2}, \quad a \in \mathbb{R}, \tag{3.6}$$

where μ_1 and μ_2 are the weights of z_1 and z_2, is a rotation.

We may then assume that the weighted degree of P is strictly less than the weighted degree of Q. Using (3.5), we obtain

$$\begin{cases} X(P)\overline{Q} + \overline{X}(\overline{Q})P = 0 \\ X(Q)\overline{P} + \overline{X}(\overline{P})Q = 0 \end{cases} \tag{3.7}$$

But (3.7) implies

$$\begin{cases} X(P) = \alpha P \\ X(Q) = \beta Q \end{cases} \tag{3.8}$$

with

$$\alpha + \overline{\beta} = 0. \tag{3.9}$$

We claim that X has no Jordan block. Indeed, putting

$$X = Y + Z, \tag{3.10}$$

where Y is the diagonal part and Z is the nilpotent part of the vecteur field X, we would obtain, since Z is also a rotation (see...), that

$$\begin{cases} Z^N(P) = \alpha^N P = 0, \\ Z^N(Q) = \alpha^N Q = 0, \end{cases} \tag{3.11}$$

which will imply that M is holomorphically degenerate. We may then write

$$X = \lambda_1 z_1 \partial_{z_1} + \lambda_1 z_2 \partial_{z_2} \tag{3.12}$$

Using (3.8), we get, for every monomial $A_{\alpha_1\alpha_2} {z_1}^{\alpha_1} {z_2}^{\alpha_2} \overline{z_1}^{\beta_1} \overline{z_2}^{\beta_2}$ of $\operatorname{Re} P\overline{Q}$,

$$\begin{cases} \lambda_1\alpha_1 + \lambda_2\alpha_2 = \alpha \\ \lambda_1\beta_1 + \lambda_2\beta_2 = -\overline{\alpha} \end{cases} \tag{3.13}$$

Suppose that $\operatorname{Re}\lambda_i \neq 0$ fore some i. Then

$$\operatorname{Re}\lambda_1 \partial_{z_1} + \operatorname{Re}\lambda_2 \partial_{z_2} \tag{3.14}$$

is also a rotation , and we obtain, using (3.13),

$$\operatorname{Re}\lambda_1(\alpha_1 + \beta_1) + \operatorname{Re}\lambda_2(\alpha_2 + \beta_2) = 0. \tag{3.15}$$

Since

$$\mu_1(\alpha_1 + \beta_1) + \mu_2(\alpha_2 + \beta_2) = 1, \tag{3.16}$$

(3.15) together with (3.16) give

$$\begin{cases} \alpha_1 + \beta_1 = m_1 \\ \alpha_2 + \beta_2 = m_2. \end{cases} \tag{3.17}$$

Suppose now that $\operatorname{Re}\lambda_i = 0, \quad i = 1, 2$. Using (3.13),

$$\begin{cases} \Lambda_1\alpha_1 + \Lambda_2\alpha_2 = -i\alpha \\ \Lambda_1\beta_1 + \Lambda_2\beta_2 = -i\alpha. \end{cases} \tag{3.18}$$

Using (3.18) and the assumption on the degree of P and Q, we get

$$\begin{cases} \Lambda_1\alpha_1 + \Lambda_2\alpha_2 = \Lambda_1\beta_1 + \Lambda_2\beta_2 \\ \mu_1\alpha_1 + \mu_2\alpha_2 < \mu_1\beta_1 + \mu_2\beta_2, \end{cases} \tag{3.19}$$

which means that the system

$$\begin{cases} \Lambda_1(\alpha_1 + \beta_1) + \Lambda_2(\alpha_2 + \beta_2) = -2i\alpha \\ \mu_1(\alpha_1 + \beta_1) + \mu_2(\alpha_2 + \beta_2) = 1 \end{cases} \tag{3.20}$$

has a unique solution

$$\begin{cases} \alpha_1 + \beta_1 = m_1 \\ \alpha_2 + \beta_2 = m_2. \end{cases} \tag{3.21}$$

The converse part is immediate. This achieves the proof of the theorem.

□

Remark 3.3. Notice that the space of rotations is at least of real dimension two.

Theorem 3.4. *Let M be given by* (2.6) *and let R be given by* (3.4). *Then*

- *If R is a complex reproducing field, M admits no generalized rotation.*
- *If R is not a complex reproducing field, M admits a generalized rotation if and only the Jacobian of the matrice*

$$\begin{pmatrix} \partial_{z_1} Q & \partial_{z_2} Q \\ \partial_{z_1} P & \partial_{z_2} P \end{pmatrix} \tag{3.22}$$

divides (in the space of homogeneous polynomials) ${Q^2}_{z_j}, \quad j = 1, 2.$

Proof. Let X be a generalized rotation. Suppose that R is a complex reproducing field for M. Then the weighted degree of P is the same as the weighted degree of Q. Applying X to $\mathsf{Im}\, w - P\overline{Q} - Q\overline{P}$, we obtain

$$X(P)\overline{Q} + X(Q)\overline{P} = 0, \tag{3.23}$$

which implies that

$$Q = \alpha P. \tag{3.24}$$

But this is impossible since M is holomorphically non degenerate. Supose now that the weighted degree of P is strictly less that the weighted degree of Q. Applying X to $\mathsf{Im}\, w - P\overline{Q} - Q\overline{P}$, we obtain

$$\begin{cases} X(Q) = 0 \\ X(P) = Q. \end{cases} \tag{3.25}$$

Putting

$$X = a_1\partial_{z_1} + a_2\partial_{z_2}, \tag{3.26}$$

where a_1 are holomorphic functions, we get, using (3.25)

$$a_1 Q_{z_1} + a_2 Q_{z_2} = 0 \tag{3.27}$$

We first consider the case where $Q_{z_1} = 0$. We obtain, using (3.27),

$$X = a_1 \partial_{z_1}. \tag{3.28}$$

Since $X(P) = Q$, we have

$$a_1 P_{z_1} = Q(z_2), \tag{3.29}$$

which implies that

$$P(z_1, z_2) = z_1 P_1(z_2) + P_2(z_2). \tag{3.30}$$

Since M is holomorphically non degenerate, it implies that $P_1(z_2)$ is a constant, and hence Q is in the ideal generated by the Jacobian J of the matrice

$$\begin{pmatrix} \partial_{z_1} Q & \partial_{z_2} Q \\ \partial_{z_1} P & \partial_{z_2} P \end{pmatrix} \tag{3.31}$$

We now consider the case where $Q_{z_1} = \mu Q_{z_2}$. Since it implies that weighted $\deg Q$ is μ_i , it is impossible by our assumption. Therefore the last case to consider is $Q_{z_1} \neq \mu Q_{z_2}$. From (3.27), we obtain

$$a_1 = -a_2 \frac{Q_{z_2}}{Q_{z_1}}. \tag{3.32}$$

By (3.25), we obtain

$$\begin{cases} a_1 Q_{z_1} + a_2 Q_{z_2} = 0 \\ a_1 P_{z_1} + a_2 P_{z_2} = Q \end{cases} \tag{3.33}$$

By Cramer's rule, we obtain

$$\begin{cases} a_1 = \dfrac{-QQ_{z_2}}{J} \\ a_2 = \dfrac{QQ_{z_1}}{J} \end{cases} \tag{3.34}$$

This achieves the proof of the theorem. □

4. Divisibility

We give a characterization of hypersurfaces of the form (1.1) which admit a generalized rotation, in terms of mutual relations of the roots of P and Q.

We first observe the following.

Lemma 4.1. *If Q_{z_j}, $j = 1, 2$ both have a root at c, then it is a root of Q as well.*

Proof. This is an immediate consequence of weighted homogeneity of Q. □

Note also that if Q has a root at c of order m, then QQ_{z_j} have a root of order $2m - 1$.

By the results of the previous section, we have

Lemma 4.2. *All the roots of $\Delta(P, Q)$ have to be roots of Q.*

Theorem 4.3. *If $P\bar{Q}$ admits a generalized rotation then, possibly after a linear change of variables, both P and Q are monomials,*

$$Q(z_1, z_2) = z_1^l z_2^m$$

and

$$P(z_1, z_2) = z_1^a z_2^b$$

where $a \leq l$ and $b \leq m$.

Proof. If the multiplicity of c as a root of Q is k and of P is m, then the multiplicity of $\Delta(P,Q)$ is $k+m-1$. On the other hand, the total degree of $\Delta(P,Q)$ is $p+q-2$. It follows that Q can have at most two distinct roots. By a linear transformation we can achieve that they coincide with the z_1 and z_2 axis. As for the roots of P, adding the power of Q only moves roots of P, but the total degree remains the same. □

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