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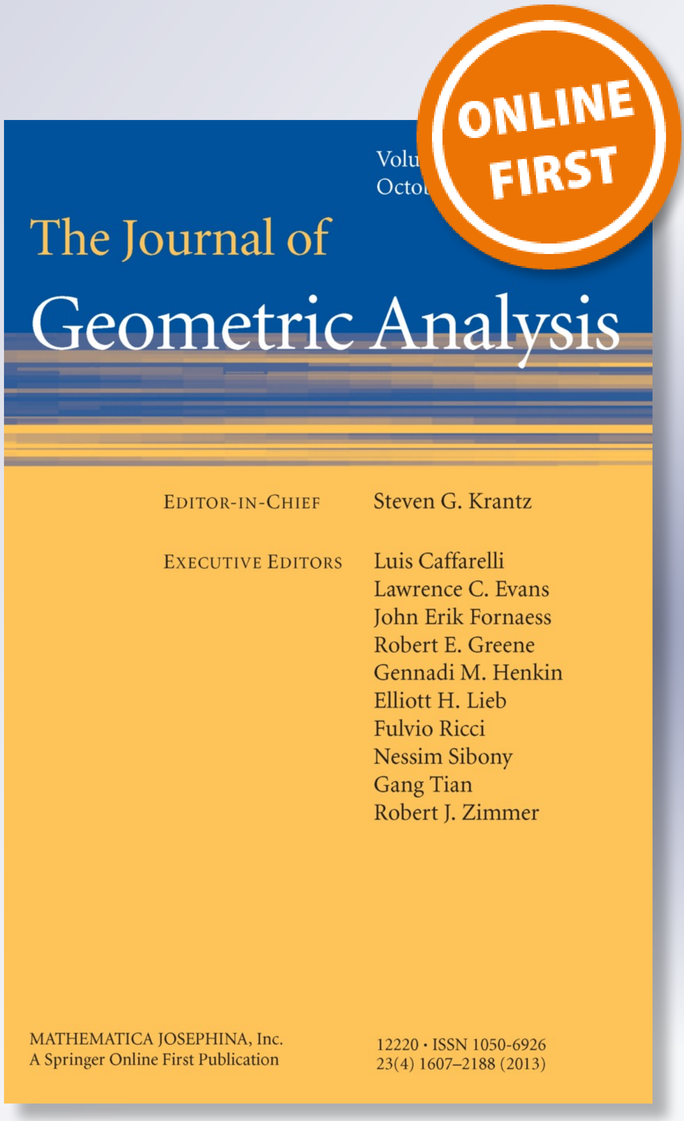

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Holomorphic Equivalence and Nonlinear Symmetries of Ruled Hypersurfaces in $\mathbb{C}^2$

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Abstract We solve the local equivalence problem for ruled hypersurfaces in $\mathbb{C}^2$ at a point of infinite type by giving a normal form. We describe all ruled hypersurfaces which admit nonlinear symmetries and give their complete characterization according to the dimension of their automorphism group. As an application, we give a bound for the dimension of the stability group for a family of nonminimal hypersurfaces.

Keywords Infinite type · Ruled hypersurfaces · Local equivalence problem · Normal forms · Nonlinear automorphisms

Mathematics Subject Classification (2010) 32V40 · 32H02

1 Introduction

The biholomorphic equivalence problem for real-analytic hypersurfaces in $\mathbb{C}^2$ dates back to an observation of Poincaré that it is—in contrast to real-analytic equivalence of such hypersurfaces—a nontrivial problem. Since his seminal work [14] this problem has played an important role in the development of complex analysis in several

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variables and also in differential geometry. The question, in its simplest form, is how to decide whether for two germs of real-analytic hypersurfaces (M, p) and (M', p') one can find a germ of a biholomorphic map $H: (\mathbb{C}^2, p) \to (\mathbb{C}^2, p')$ which maps M to M'.

For *Levi-nondegenerate* hypersurfaces in $\mathbb{C}^2$, the problem has been solved by Élie Cartan [3, 4]; this has been later generalized to Levi-nondegenerate hypersurfaces in $\mathbb{C}^N$, $N \geq 2$ by Tanaka [15] and Chern–Moser [5]. Levi-degenerate hypersurfaces have received less attention, though especially in applications, they play a prominent role. In $\mathbb{C}^2$, Levi-degenerate hypersurfaces of finite type (whose analytical importance goes back to work of Kohn [8]) have been classified by the first author [9].

One of the main consequences of this classification is that the only hypersurfaces which possess nonlinearizable symmetries, or an automorphism group of high enough dimension, are the ball and its weakly pseudoconvex analogues given by $t = |z|^{2k}$. This follows from work of Kruzhilin and Loboda [13] for strictly pseudoconvex hypersurfaces and from [10] in the finite type case, and for the dimension of automorphisms by the work of Beloshapka [1].

In this paper we start a systematic study of the remaining case of infinite type hypersurfaces in $\mathbb{C}^2$ and give a complete normal form for hypersurfaces which are 1-nonminimal. Using coordinates $(z, w) \in \mathbb{C}^2$ with $z = x + iy$, $w = s + it$, these are given by an equation of the form

$$t = s\psi(z, \bar{z}, s),$$

where $\psi(z, \bar{z}, 0) \neq 0$.

Before stating the normal form result, let us first consider the class of *ruled hypersurfaces* of infinite type, which play the role of "model" hypersurfaces for the classification problem for the 1-nonminimal hypersurfaces.

Definition 1 A germ of a real-analytic hypersurface $(M, 0)$ is a *ruled* hypersurface if there exist coordinates $(z, w) \in \mathbb{C}^2$ such that M is given by an equation of the form

$$t = sA(z, \bar{z}).$$

The set of all such germs is denoted by $\mathcal{A}$.

Besides solving the equivalence problem in $\mathcal{A}$, we determine which hypersurfaces support nonlinearizable symmetries. We show that there are surprisingly many examples of such hypersurfaces in $\mathcal{A}$, some of them of a very different nature from the ones which have previously been known. Furthermore, we completely characterize the possible symmetries explicitly and give a list of all the hypersurfaces with high-dimensional symmetry groups, exhibiting those as "blowups" of spheres or, more generally, higher-dimensional hyperquadrics.

In contrast to the case of *finite type* hypersurfaces, the condition of being of infinite type is (except for the trivial case of a Levi-flat hypersurface) not an open condition: the set of points of infinite type in a hypersurface $M \subset \mathbb{C}^2$ form (locally) a complex-analytic hypersurface $E \subset M$, the so-called *exceptional hypersurface*.

One approach to the biholomorphic equivalence problem in $\mathcal{A}$ is to find particular representatives of the equivalence classes of $\mathcal{A}$, the *normal forms*. The complete normal form for $\mathcal{A}$ is described in Sect. 3.

The next question is which *nonlinear* maps hypersurfaces in normal form actually allow. We show that there are essentially 3 families of ruled hypersurfaces which possess any nonlinear maps. One can think about these families, denoted by $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$, as "primitive" models in a sense to be made precise below. Our blowups are an analytic procedure, i.e., the actual preimages are obtained by an application of the implicit function theorem; it is a nontrivial fact that this is possible, which will be demonstrated later in the paper.

- $\mathcal{A}_1$ consists of hypersurfaces which are preimages of Levi-nondegenerate hyperquadrics in $\mathbb{C}^{2k-1}$, given by $\operatorname{Im}\eta = \operatorname{Re}(\sum_{j=1}^{k-1} p_j\zeta_j\overline{\xi_j})$ (where $p_j = \overline{p_{k-j}}$), under the map
$$\eta = w, \qquad \zeta_j = z^j w^{(1-\frac{j}{k})}, \qquad \xi_j = z^{k-j} w^{\frac{j}{k}}.$$
We call the elements of $\mathcal{A}_1$ *rational blowups of hyperquadrics* (even though the term "blowup" is used in a different sense from the usual). A hypersurface in $\mathcal{A}_1$ is completely determined by k and $p_1, \dots, p_{\lfloor (k-1)/2 \rfloor}$. Other characterizations can be found in Lemma 10.
- $\mathcal{A}_2$ consists of hypersurfaces which are preimages of a circular finite type hypersurface in $\mathbb{C}^2$ given by $\operatorname{Im}\eta = |\zeta|^{2\ell}$ under a map of the form
$$\zeta = z w^{\frac{1}{2\ell} + iT}, \qquad \eta = \pm w,$$
where $T \in \mathbb{R}$. $\mathcal{A}_2$ are *transcendental blowups of the ball* if $T \neq 0$. More descriptions of $\mathcal{A}_2$ can be found in Lemma 11.
- $\mathcal{A}_3$ is of a different nature, and we describe its elements as *tube-like hypersurfaces*. A hypersurface given by an equation $t = sA(z, \bar{z})$ in $\mathcal{A}_3$ is uniquely determined by the fact that it possesses an infinitesimal automorphism of the form $w^2 \frac{\partial}{\partial z}$, and also by a part of the Taylor expansion of its defining function $A(z, \bar{z})$, namely, $A(z, 0)$. The main difference here is that in contrast to $\mathcal{A}_1$ and $\mathcal{A}_2$, the parameter space is still *infinite dimensional*. The class $\mathcal{A}_3$ is first described in Lemma 12 and discussed in detail in Sect. 6.

Let us also say that a hypersurface $\tilde{t} = \tilde{s}\tilde{A}(z, \bar{z})$ is a *root* of a hypersurface $t = sA(z, \bar{z})$ if it is obtained from the latter by a modification of the form $\tilde{w}^k = w$. We can now finally state the following theorem.

Theorem 1 *Let M be a ruled hypersurface in $\mathbb{C}^2$, $p \in M$ of infinite type. If (M, p) allows an automorphism which is nonlinear in normal coordinates, then M has a root in $\mathcal{A}_1$, $\mathcal{A}_2$, or $\mathcal{A}_3$.*

We also give a complete characterization of the infinite type ruled hypersurfaces whose automorphism group has dimension exceeding 1. We give a complete description of the possible dimensions of the automorphism groups $\operatorname{Aut}(M, p)$ (which are known to be finite-dimensional Lie groups by [6]), together with a complete list of possible hypersurfaces for each dimension.

In order to write down the table, we observe that the Lie algebra of $\mathrm{Aut}(M, p)$, which is traditionally denoted by $\mathfrak{hol}(M, p) =: \mathfrak{g}$, is graded in a natural way by

$$\mathfrak{g}_k = \left\{\alpha(z)w^k \frac{\partial}{\partial z} + \beta(z)w^{k+1}\frac{\partial}{\partial w} \in \mathfrak{hol}(M, p)\right\},$$

and that we necessarily have a splitting $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_m \oplus \mathfrak{g}_{2m}$ for some $m \in \mathbb{N}$ (see Sect. 2.2; the latter assertion is nontrivial).

One of the basic subgroups, the isotropy subgroup of the automorphism group, becomes nearly useless in our setting. Even though $\mathfrak{g}_m$ and $\mathfrak{g}_{2m}$ are always in the isotropy part, the nature of the PDE fulfilled by the defining equation coming from an infinitesimal automorphism in $\mathfrak{g}_m$ or $\mathfrak{g}_{2m}$ is dictated by whether $\alpha(0) = 0$ or $\alpha(0) \neq 0$. This reflects the fact that any holomorphic map $H\colon M \to M'$ has to satisfy $H(E) \subset E'$. Let us therefore define the subspace

$$\mathfrak{g}_k^0 = \left\{\alpha(z)w^k \frac{\partial}{\partial z} + \beta(z)w^{k+1}\frac{\partial}{\partial w} \in \mathfrak{g}_k : \alpha(0) = 0\right\}.$$

We state our result formally as

Theorem 2 *Assume that* $(M, 0)$ *is a ruled hypersurface of infinite type in* $\mathbb{C}^2$. *If* $\dim \mathrm{Aut}(M, 0) > 1$, *then* $(M, 0)$ *is locally equivalent to one of the hypersurfaces in Table* 1.

Empty entries mean unspecified dimensions. Formulas are given for φ such that any hypersurface with those dimensions is equivalent to one of the form

$$t = s\tan\left(\frac{\varphi(z, \bar{z})}{2}\right).$$

We should stress that every ruled hypersurface whose space of infinitesimal automorphisms has dimension at least 3 is a modification of a sphere in $\mathbb{C}^2$. While this might not be apparent from the table above at first glance, we will show that each of the last six entries can be obtained as the preimage of the sphere under an explicitly given map. The class $\mathcal{A}_1$ corresponds to the fourth line, the class $\mathcal{A}_3$ to the third line, and the class $\mathcal{A}_2$ to the fifth line of the table.

Let us note that our results here also illustrate the results of Kossovskiy and Shafikov about analytic continuation of maps between nonminimal hypersurfaces [11]. Another context in which the manifolds considered and characterized here appear is the recent work of Kim and Thu [7] motivated by the Greene–Krantz conjecture.

The importance of the result in Theorem 2 is that it allows us to give a very refined normal form for the class of 1-nonminimal hypersurfaces mentioned above. Let us write

$$\mathcal{F} = \left\{\psi(z, \bar{z}, s) \colon \psi(z, \bar{z}, 0) = 0,\ \psi_s(z, \bar{z}, 0) \neq 0\right\},$$

Table 1 Dimensions of automorphism groups, defining equation $t = s\tan(\varphi/2)$

$\mathfrak{g}$	$\mathfrak{g}_0$	$\mathfrak{g}_0^0$	$\mathfrak{g}_m$	$\mathfrak{g}_m^0$	$\mathfrak{g}_{2m}$	$\mathfrak{g}_{2m}^0$	Defining equation	Satisfying
≥ 2	2	2					$\varphi(\lvert z\rvert^2)$	
	2	1					$\varphi(x)$	
					1	0	$\frac{\varphi(x,y)}{m}$	$\varphi_x + \varphi_y\tan(\varphi) = 0$
					1	1	$\frac{\varphi(x,y)}{m}$	$k\sin(\varphi) = (x\varphi_x + y\varphi_y)\cos(\varphi) + (x\varphi_y - y\varphi_x)\sin(\varphi)$ $e^{2i\varphi} - 1 = p(z, e^{\frac{2i\varphi}{k}}\bar{z}) + \bar{p}(e^{\frac{2i\varphi}{k}}\bar{z}, z)$
≥ 3	2	2	1	1	0	0	$\frac{\varphi(\lvert z\rvert^2)}{m}$	$u\varphi'(u) = \frac{k\tan(\varphi(u))}{1+T\tan(\varphi(u))}$, $2 \leq k \in \mathbb{N}$, $T \in \mathbb{R}$ $\sin(\varphi(u)) = u^k e^{-kT\varphi(u)}$
	2	1	1	0	0	0	$\frac{\varphi(x,y)}{m}$	$\sin(\varphi)(R-x) + \cos(\varphi)(1-y) = e^{-T\varphi}$, $R, T \in \mathbb{R}$, $R + T \neq 0$
4	4	2	0	0	0	0	$\frac{\ln(1 - C\lvert z\rvert^2)}{mC}$	$0 \neq C \in \mathbb{R}$
	4	2	0	0	0	0	$\frac{\lvert z\rvert^2}{m}$	in the limiting case $C = 0$
5	2	2	2	0	1	1	$\frac{\arcsin(\lvert z\rvert^2)}{m}$	
	2	1	2	0	1	0	$\frac{1}{2im}\ln(\frac{i+\sqrt{\lvert R-i-z\rvert^2-1}}{R+i-\bar{z}})$	$R \neq 0$

and decompose

$$\mathcal{F} = \bigoplus_{j=1}^{\infty} \mathcal{F}_j,$$

where $\mathcal{F}_j$ consists of the power series which are multiples of s^j. We use the weight 0 for z, and the weight 1 for w; i.e., power series in $\mathcal{F}_j$ are (weighted) homogeneous of degree j.

After a preliminary normalization, we can assume that the defining equation $t = \psi(z, \bar{z}, s)$ of any given 1-nonminimal hypersurface has its lowest-order term (which defines a ruled hypersurface) in normal form for ruled hypersurfaces. For any of these initial terms (which we will denote by ψ_1) we can find a normal form as follows.

Theorem 3 *Let $A(z, \bar{z})$ be in normal form for ruled hypersurfaces, and let $\mathfrak{g} = \mathfrak{hol}(\{t = sA(z, \bar{z})\}, 0)$. Then there exists a subspace $\mathcal{N}_A \subset \bigoplus_{j\geq 2} \mathcal{F}_j$ such that the following holds: For every 1-nominimal (formal) hypersurface M, defined by an equation $t = sA(z, \bar{z}) + \psi(z, \bar{z}, s)$ with $\psi_s(z, \bar{z}, 0) = 0$ and for any $X \in \mathfrak{g}$ there exists a uniquely determined biholomorphism $H = H_X$ which brings M into normal form, i.e., $(H)^{-1}(M)$ is given by an equation*

$$t = sA(z, \bar{z}) + \tilde{\psi}(z, \bar{z}, s)$$

with $\tilde{\psi}(z, \bar{z}, s) \in \mathcal{N}_A$. Two germs of real-analytic 1-nonminimal hypersurfaces are biholomorphically equivalent if and only if they can be given by identical formal normal forms.

An immediate consequence of Theorem 3 is a dimension bound for automorphism groups of 1-nonminimal hypersurfaces; i.e., $\dim \operatorname{Aut}(M, p) \leq 5$. However, hypersurfaces whose automorphism groups are of maximal dimension can also be characterized. This generalizes a result of Beloshapka [2].

Theorem 4 *Let $(M, 0)$ be a germ of a 1-nonminimal hypersurface in $\mathbb{C}^2$. Then* $\dim \operatorname{Aut}(M, 0) \leq 5$. *If* $\dim \operatorname{Aut}(M, 0) = 5$, *then $(M, 0)$ is biholomorphically equivalent to one of the hypersurfaces in the last two lines of Table 1.*

Let us give an outline of this paper. For the proof of Theorem 1, we show in Sect. 4 that the study of nonlinear automorphisms reduces to the study of the Lie algebra of infinitesimal CR automorphisms (in our setting, we give a simple proof of this fact due to the presence of a distinguished 1-parameter group of infinitesimal automorphisms, which can be used to generate an infinitesimal automorphism from any automorphism). We also characterize completely the infinitesimal CR automorphisms which can arise. In Sect. 5 we characterize in several ways the exceptional hypersurfaces supporting these infinitesimal automorphisms. Let us point out that we also encounter the examples given earlier by Kowalski [12] and Zaitsev [16] which appear as the second-to-last line in Table 1.

We next study hypersurfaces which allow at least one "special" infinitesimal automorphism; we refer to them as "tube-like" hypersurfaces or simply *tubes* (they might not be tubes in the traditional sense, though). This is started in Sect. 6. The first family of such tubes, which we refer to as *degenerate tubes*, is studied more closely in Sect. 7, where we show that they support *exactly* two infinitesimal automorphisms. The nondegenerate tubes need a more careful study—they are the only ones whose symmetry group has dimension greater than two. We first study nondegenerate tubes with "many" infinitesimal automorphisms of homogeneity zero in Sect. 8. We then give a thorough discussion of sphere blowups in Sect. 9. Finally, in Sect. 10 we show that any hypersurface having "enough" (i.e., more than 2 linearly independent infinitesimal automorphisms, not all of zero homogeneity) is biholomorphically equivalent to a sphere blowup. The lemmas in this section complete the classification in the above table, and the proof of Theorem 2 is given in the end. In Sect. 11 we finally give the classification result in Theorem 3.

We start in Sect. 2 by introducing our notation and the different forms of defining functions which we shall use.

2 Notation and Preliminaries

2.1 Defining Equations

Let (M, p) be a germ of a ruled, real-analytic (or formal) hypersurface in $\mathbb{C}^2$. We are interested in the case where p is a point of infinite type on M. Then there is a complex hypersurface E through p contained in M, and the real lines making up M intersect E transversally.

We choose local coordinates (z, w) in $\mathbb{C}^2$ in which $p = 0$ and such that locally around p, $E = \{w = 0\}$. M, in these coordinates, is given by an equation of the form

$$t = s\, A(z, \bar{z}), \tag{1}$$

or equivalently,

$$w = \bar{w} B(z, \bar{z}) = \bar{w} \frac{1 + i A(z, \bar{z})}{1 - i A(z, \bar{z})}, \tag{2}$$

where $A(z, \chi)$ is a germ of a holomorphic function at $(0, 0) \in \mathbb{C}^2$. We will assume that $A(0, 0) = 0$, or equivalently, that $B(0, 0) = 1$. We will write $B(z, \chi) = e^{i\varphi(z,\chi)}$. We will refer to B given by (2) as the B associated with A. Note that $A = \tan(\varphi/2)$.

Remark 1 (Reality conditions) We note that (1) or (2) defines a *real* hypersurface in $\mathbb{C}^2$ if and only if $A(z, \chi) = \bar{A}(\chi, z)$ or equivalently, if $B(z, \chi)\bar{B}(\chi, z) = 1$, which again is equivalent to $\varphi(z, \chi) = \bar{\varphi}(\chi, z)$; for a power series f, we denote by $\bar{f}$ the series where the coefficients have been replaced by their complex conjugates.

Remark 2 (Basic symmetry) Note that in these coordinates, the vector field $w\frac{\partial}{\partial w}$ generates a 1-parameter group of transformations of M given by $(z, w) \mapsto (z, tw)$ for $t \neq 0$. All of our ruled hypersurfaces thus support an infinitesimal symmetry of this form.

Our first goal is to find a particularly simple form for the defining equation. To this end, we start by removing the "harmonic terms" and observe that after a change of coordinates of the form $(z', w') = (z, e^{-i\varphi(z,0)} w)$ we can assume furthermore that

$$A(z,0) = A(0,\chi) = 0, \quad \text{or equivalently,} \quad B(z,0) = B(0,\chi) = 1; \tag{3}$$

if $H(z,w) = (f(z,w), g(z,w))\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a germ of a biholomorphic map, and $H(M)$ is given by $t = sA'(z,\bar{z})$, then $A'(z,0) = A'(0,\chi) = 0$ (i.e., H respects the normalization (3)) if and only if $g(z,0) = 0$ and $g_w(z,0) = g_w(0,0)$.

We now observe that if A satisfies (3), it transforms in a very simple way under the action of biholomorphisms.

Lemma 2 *Assume that $(M,0)$ is given by (1) or (2) and that it satisfies (3). Let $(M',0)$ be likewise given by $t = s\,A'(z,\bar{z})$, with $A'(z,0) = A'(0,\bar{z}) = 0$. If $H = (f,g)\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ is a germ of a holomorphic map sending $(M,0)$ into $(M',0)$ which satisfies $g_w(0) \neq 0$, then*

$$A'\big(f(z,0), \bar{f}(\chi,0)\big) = A(z,\chi). \tag{4}$$

Proof We start with the basic transformation equation, writing it in terms of the associated B and B':

$$g\big(z, \tau B(z,\chi)\big) = \bar{g}(\chi,\tau) B'\big(f\big(z, \tau B(z,\chi)\big), \bar{f}(\chi,\tau)\big). \tag{5}$$

First note that $g(z,0) = 0$. Taking the derivative with respect to τ and evaluating at $\tau = 0$, we see that $g_w(z,0)B(z,\chi) = \bar{g}_\tau(\chi,0)B'(f(z,0), \bar{f}(\chi,0))$. Since $B(z,0) = B'(z,0) = 1$, $g_w(z,0) = \bar{g}_\tau(0)$, and so $B(z,\chi) = B'(f(z,0), \bar{f}(\chi,0))$, which is equivalent to (4). □

2.2 Decomposition of the Space of Infinitesimal Automorphisms

Since we are dealing with hypersurfaces whose defining equation is of the form $t = sA(z,\bar{z})$, the defining equation is homogeneous if we define the weight of z to be 0 and the weight of w to be 1. The Euler vector field corresponding to this choice of weights is just our basic symmetry from Remark 2, $T = w\frac{\partial}{\partial w}$, and we can decompose the space of holomorphic vector fields $\alpha(z,w)\frac{\partial}{\partial z} + \beta(z,w)\frac{\partial}{\partial w}$ with $\beta(z,0) = 0$ in the eigenspaces of $\operatorname{ad} T$, i.e.,

$$\begin{aligned}
X &= \alpha(z,w)\frac{\partial}{\partial z} + \beta(z,w)\frac{\partial}{\partial w} \\
&= \Big(\sum_{j\geq 0} \alpha_j(z) w^j\Big)\frac{\partial}{\partial z} + \Big(\sum_{j\geq 0} \beta_j(z) w^{j+1}\Big)\frac{\partial}{\partial w} \\
&= \sum_{j\geq 0}\Big(\alpha_j(z) w^j \frac{\partial}{\partial z} + \beta_j(z) w^{j+1}\frac{\partial}{\partial w}\Big) \\
&= \sum_{j\geq 0} X_j.
\end{aligned}$$

Recall that $\mathfrak{hol}(M,0)$ denotes the Lie algebra of infinitesimal real-analytic CR automorphisms of $(M,0)$, i.e., germs of holomorphic vector fields $X=\alpha(z,w)\frac{\partial}{\partial z}+\beta(z,w)\frac{\partial}{\partial w}$ such that $\operatorname{Re} X$ is tangent to M near 0. We can thus also decompose $\mathfrak{hol}(M,0)=:\mathfrak{g}(M)$ into homogeneous parts and write

$$\mathfrak{g}(M)=\sum_{j\geq 0}\mathfrak{g}_j(M).$$

3 The Normalization Procedure

In this section, we are going to give an algorithm for deciding if two ruled hypersurfaces are biholomorphically equivalent. From the last section, we recall that we can identify the space of ruled hypersurfaces which are not Levi-flat with the following space of power series:

$$\mathcal{A}=\left\{A\in\mathbb{C}[[z,\bar{z}]]\colon A(z,\bar{z})=\bar{A}(\bar{z},z),\, A\neq 0,\, A(z,0)=A(0,\bar{z})=0\right\}. \tag{6}$$

We are going to abuse notation and identify a ruled hypersurface M given by the equation $t=sA(z,\bar{z})$ with $A\in\mathcal{A}$ and simply say "the hypersurface A".

We would like to decide whether two elements of $\mathcal{A}$ belong to the same orbit in $\mathcal{A}$ under the action of germs of biholomorphisms (preserving the special form of defining functions in $\mathcal{A}$). We will do so by exhibiting a unique element of each such orbit, which will be referred to as the "distinguished normal form".

Let us order points in $\mathbb{N}^2$ lexicographically: we write $(a,b)\leq_{\text{lex}}(c,d)$ if either $b<d$ or if $b=d$ and $a\leq c$. For such an $A(z,\chi)=\sum_{\alpha,\beta}A_{(\alpha,\beta)}z^\alpha\chi^\beta\in\mathcal{A}$, we define

$$\operatorname{in} A=\min\left\{(\alpha,\beta)\in\mathbb{N}^2\colon A_{(\alpha,\beta)}\neq 0\right\},\quad \text{and}\quad \operatorname{it} A=A_{\operatorname{in} A} \tag{7}$$

where the minimum is with respect to the lexicographic order just introduced. We note that Lemma 2 implies that the preceding definition is indeed independent of the choice of coordinates, and thus gives rise to a local invariant of $(M,0)$. The normalization of $A\in\mathcal{A}$ is now done by a change of coordinates of the form $(z',w')=(\psi(z),w)$ and reduces the equivalence problem in $\mathcal{A}$ to a linear question:

Lemma 3 *Let $A\in\mathcal{A}$ be given. Then there exists a holomorphic function ψ with $\psi'(0)\neq 0$ such that in the coordinates $(z',w')=(\psi(z),w)$, the hypersurface corresponding to A is given by $A'(z',\chi')$ satisfying*

$$A'(z',0)=A'(0,\chi)=0,\quad A'_{\operatorname{in} A+(0,k)}=0,\quad k>0,\quad \operatorname{it} A'=\begin{cases}1 & \operatorname{in} A\neq(n,n),\\ \pm 1 & \operatorname{in} A=(n,n).\end{cases} \tag{8}$$

We shall say that A' is in normal form if it satisfies (8). If A and A' are both in normal form, and $H=(f(z,w),g(z,w))$ is a germ of a biholomorphism which transforms A into A', then $H=L\circ T$, where L is a linear map of the form $(z,w)\mapsto(\lambda z,w)$ with $|\lambda|=1$ and T is an automorphism of the hypersurface corresponding to A.

Proof We write $A(z,\chi)=\sum_{\gamma\in\mathbb{N}} A_\gamma(z)\chi^\gamma$; if $(\alpha,\beta):=$ in A, then we have $A_\beta(z)=(\varphi(z))^\alpha$ where $\varphi'(0)\neq 0$. We thus define $z'=\varphi(z)$, and write $z=\psi(z')$. By Lemma 2, $A(\psi(z'),\bar{\psi}(\chi'))=A'(z',\chi')$. We thus have $A'_\beta(z')=A_\beta(\psi(z'))\bar{\psi}'(0)^\beta=z'^\alpha\bar{\psi}'(0)^\beta$, as required. Applying a map of the form $(z,w)\mapsto(\lambda z,w)$ with some $\lambda\in\mathbb{C}$, we can achieve that it $A'=1$ unless $\alpha=\beta$. If on the other hand $\alpha=\beta$, then it $A'\in\mathbb{R}$ by the reality conditions, and we rescale with $\lambda>0$ to obtain it $A'=\pm 1$.

If A and A' are both in normal form and $H(z,w)$ is any biholomorphism mapping A to A', Lemma 2 implies that $A'(f(z,0),\bar{f}(\chi,0))=A(z,\chi)$. The preceding computation now shows that $f(z,0)=\lambda z$ for some constant $\lambda\neq 0$ satisfying $\lambda^\alpha\bar{\lambda}^\beta=1$. If we consider the map $S\colon (z,w)\mapsto(\lambda^{-1}z,w)$, another application of Lemma 2 shows that $S\circ H$ maps A to itself. Thus, $H=L\circ T$ where $L=S^{-1}$ and T is an automorphism of A. □

In particular, if A and A' are biholomorphically equivalent, they are already equivalent under a linear transformation of the form $(z,w)\mapsto(\lambda z,w)$. This means that A and A', both in normal form, define biholomorphically equivalent hypersurfaces if and only if there exists λ with $|\lambda|=1$ such that for all $(\alpha,\beta)\in\mathbb{N}^2$,

$$A'_{(\alpha,\beta)}\lambda^\alpha\bar{\lambda}^\beta=A_{(\alpha,\beta)}. \tag{9}$$

Let us define for $A\in\mathcal{A}$ a set $F(A)\subset\mathbb{N}^2$ of A by

$$F(A)=\left\{(\alpha,\beta)\in\mathbb{N}^2\colon A_{(\alpha,\beta)}\neq 0\right\}, \tag{10}$$

which is a biholomorphic invariant by the preceding observations. Next we define $S(F)$ for a subset $F\in\mathbb{N}^2$ by

$$S(F)=\gcd\left\{\beta-\alpha\colon(\alpha,\beta)\in F\right\}, \tag{11}$$

where we adopt the convention that $\gcd(0)=0$. For $S(F(A))$ we shall simply write $S(A)$.

For any $F\subset\mathbb{N}^2$, also define a finite sequence $N(F)=((\alpha_1,\beta_1),\dots,(\alpha_d,\beta_d))$ with $(\alpha_j,\beta_j)\in F$ inductively as follows: First, we set $(\alpha_1,\beta_1)=\min\{f\in F\}$ (as usual, in the lexicographic sense). If (α_k,β_k) has been defined for $k\leq j$, we define $(\alpha_{j+1},\beta_{j+1})$ by

$$\begin{aligned}(\alpha_{j+1},\beta_{j+1})=\min\Big\{&(\alpha,\beta)\in F\colon(\alpha_j,\beta_j)\leq_{\text{lex}}(\alpha,\beta),\\ &\gcd\big(f\in F\colon f\leq_{\text{lex}}(\alpha,\beta)\big)\neq\gcd(\beta_k-\alpha_k\colon k\leq j)\Big\}.\end{aligned}$$

This process stops with a finite sequence $N(F)$ which, if we write $e_j=\gcd(\beta_k-\alpha_k\colon k\leq j)$ satisfies that

$$S(F)=e_d|e_{d-1}|\cdots|e_1.$$

We now refine the normalization conditions on A as follows. The normalization in Lemma 3 corresponds to choosing a λ_1 such that $\lambda_1^{e_1}(\text{it}\,A)\in\{\pm 1\}$. A linear transformation of the form $(z,w)\mapsto(\lambda z,w)$ respects this normalization if and only if $\lambda^{e_1}=1$. Next, we consider $A_{(\alpha_2,\beta_2)}$; under the action of the rotations $z\mapsto\lambda z$ with

$\lambda^{e_1} = 1$, $A_{(\alpha_2,\beta_2)}$ necessarily has nontrivial orbit. We choose λ_2 with $\lambda_2^{e_1} = 1$ such that $\arg A_{(\alpha_2,\beta_2)}\lambda_2^{e_2} = \min\{\arg A_{(\alpha_2,\beta_2)}\lambda^{e_2} : \lambda^{e_1} = 1\}$. Proceeding inductively, we produce a linear map $(z, w) \mapsto (\lambda z, w)$ such that in these coordinates, the defining function A satisfies

$$\arg A_{(\alpha_j,\beta_j)} = \min\bigl\{\arg A_{(\alpha_j,\beta_j)}\lambda^{e_j} : \lambda^{e_{j-1}} = 1\bigr\}.$$

We shall say that such an A is in *distinguished normal form*. By construction, the only linear maps respecting these normalizations leave A invariant. We have arrived at a complete normal form for hypersurfaces in $\mathcal{A}$:

Theorem 5 *Let $A \in \mathcal{A}$, $A' \in \mathcal{A}$ be in distinguished normal form. Then the hypersurfaces described by A and A' are biholomorphically equivalent if and only if $A(z,\chi) = A'(z,\chi)$. Furthermore, the linear automorphisms of A are given by $(z, w) \mapsto (\lambda z, tw)$ where $t \in \mathbb{R}$, $|\lambda| = 1$, and $\lambda^{S(A)} = 1$.*

4 Nonlinear Automorphisms

In the preceding section, we have introduced the distinguished normal form, which completely solves the biholomorphic equivalence problem in $\mathcal{A}$. We shall use this knowledge to characterize hypersurfaces $A \in \mathcal{A}$ which allow nonlinear automorphisms. Our starting point is the observation that we can essentially restrict ourselves to studying infinitesimal automorphisms. This is true since the hypersurfaces we consider here are homogeneous with respect to the weights introduced in Sect. 2.2:

Lemma 4 *Let A be a ruled hypersurface and assume that*

$$H(z, w) = (f(z, w), g(z, w))$$

is an automorphism of A such that

$$f(z, w) = z + \alpha(z)w^\ell + O\bigl(w^{\ell+1}\bigr), \qquad g(z, w) = w + \beta(z)w^{\ell+1} + O\bigl(w^{\ell+2}\bigr),$$

with α and β not both identically vanishing. Then the vector field

$$X_H = \alpha(z)w^\ell \frac{\partial}{\partial z} + \beta(z)w^{\ell+1}\frac{\partial}{\partial w}$$

has its real part tangent to A and thus generates a 1-parameter group of holomorphic automorphisms of A.

Proof We write A in terms of its complex defining function B and as for (5), arrive at

$$g\bigl(z, \tau B(z,\chi)\bigr) = \bar{g}(\chi, \tau)B\bigl(f\bigl(z, \tau B(z,\chi)\bigr), \bar{f}(\chi, \tau)\bigr).$$

Comparing the coefficient of $\tau^{\ell+1}$ on both sides, we arrive at

$$\beta(z)B(z,\chi)^{\ell+1} = \bar{\beta}(\chi)B(z,\chi) + \alpha(z)B_z(z,\chi)B(z,\chi)^\ell + \bar{\alpha}(\chi)B_\chi(z,\chi),$$

which in turn is equivalent to $\operatorname{Re} X_H$ being tangent to the hypersurface defined by B, since

$$(X_H + \bar{X}_H)\big(w - \tau B(z,\chi)\big) = \beta(z)w^{\ell+1} - \alpha(z)w^{\ell}\tau B_z(z,\chi) - \bar{\beta}(\chi)\tau^{\ell+1}B(z,\chi) - \bar{\alpha}(\chi)\tau^{\ell+1}B_\chi(z,\chi). \qquad \square$$

We shall also abuse notation and refer to a holomorphic vector field whose real part is tangent to A as an infinitesimal CR automorphism of A.

In fact, we can simplify computations somewhat by observing that if A defines a hypersurface which allows for a nonlinear automorphism H as in Lemma 4, then there is a hypersurface $\tilde{A} \in \mathcal{A}$ which allows for a closely related automorphism for which $\ell = 1$:

Lemma 5 *Let A be a ruled hypersurface and assume that*

$$H(z,w) = (f(z,w), g(z,w))$$

is an automorphism of A such that

$$f(z,w) = z + \alpha(z)w^{\ell} + O\big(w^{\ell+1}\big), \qquad g(z,w) = w + \beta(z)w^{\ell+1} + O\big(w^{\ell+2}\big),$$

with α and β not both identically vanishing. Let $T(z,w) = (z, w^{\ell})$. Then there exists a ruled hypersurface $\tilde{A}$ such that $T(A) \subset \tilde{A}$ which has an automorphism $\tilde{H}(z,w) = (\tilde{f}(z,w), \tilde{g}(z,w))$ where

$$\tilde{f}(z,w) = z + \alpha(z)w + O\big(w^2\big), \qquad \tilde{g}(z,w) = w + \beta(z)w^2 + O\big(w^3\big).$$

Proof We again use the representation of A by B. Define $\tilde{B} = B^{\ell}$. Now if $(z,w) \in A$, then $w = \bar{w}B(z,\bar{z})$, and so $(z, w^{\ell}) \in \tilde{A}$. The proof is completed by computing that $X_{\tilde{H}} = \alpha(z)w\frac{\partial}{\partial z} + \ell\beta(z)w^2\frac{\partial}{\partial w}$ has real part tangent to $\tilde{A}$. $\square$

We thus start by looking for hypersurfaces which have an infinitesimal automorphism of the form $X = \alpha(z)w\frac{\partial}{\partial z} + \beta(z)w^2\frac{\partial}{\partial w}$. We will first discuss the case where $\alpha(0) = 0$.

Lemma 6 *Assume $X = \alpha(z)w\frac{\partial}{\partial z} + \beta(z)w^2\frac{\partial}{\partial w}$ is an infinitesimal CR automorphism of A (given in normal form) which satisfies $\alpha(0) = 0$. Write* in $A = (m,n)$. *Then, after possibly rescaling X,*

$$X = \left(\frac{1}{n+m} + iT\right)zw\frac{\partial}{\partial z} + w^2\frac{\partial}{\partial w},$$

where $T = 0$ unless $n = m$.

Proof Since X is tangent to A, if we write the equation in terms of B, we have that

$$\beta(z)B(z,\chi)^2 = \bar{\beta}(\chi)B(z,\chi) + \alpha(z)B_z(z,\chi)B(z,\chi) + \bar{\alpha}(\chi)B_\chi(z,\chi).$$

Evaluating at $\chi = 0$, we see that $\beta(z) = \bar{\beta}(0)$; thus $\beta(z)$ is equal to a nonzero real constant, and we rescale X so that $\beta = 1$. We then have the equation

$$B(z,\chi)^2 = B(z,\chi) + \alpha(z)B_z(z,\chi)B(z,\chi) + \bar{\alpha}(\chi)B_\chi(z,\chi);$$

comparing the coefficient of χ^n on both sides of this equation leads to

$$z = m\alpha(z) + n\bar{\alpha}'(0)z,$$

so α is linear, i.e., $\alpha(z) = z\alpha'(0)$. Furthermore, comparing real and imaginary parts of the last equation, we see that $(n+m)\operatorname{Re}\alpha'(0) = 1$ and unless $m = n$, $\operatorname{Im}\alpha'(0) = 0$. □

In order to discuss the remaining case $\alpha(0) \neq 0$, we first notice that only hypersurfaces satisfying in $A = (1, m)$ can support such an infinitesimal automorphism:

Lemma 7 *If $X = \alpha(z)w\frac{\partial}{\partial z} + \beta(z)w^2\frac{\partial}{\partial w}$ is an infinitesimal CR automorphism of A (given in normal form) which satisfies $\alpha(0) \neq 0$, then* in $A = (1, m)$.

Proof As in the start of the proof of the preceding Lemma 6, we examine the equation

$$\beta(z)B(z,\chi)^2 = \bar{\beta}(\chi)B(z,\chi) + \alpha(z)B_z(z,\chi)B(z,\chi) + \bar{\alpha}(\chi)B_\chi(z,\chi).$$

If $\alpha(0) \neq 0$ and in $A = (m, n)$, the last term on the right-hand side contains a term with χ^{n-1}, which cannot be cancelled by any other term unless $n = 1$. □

For such vector fields, we have two different formulas according to whether $m = 1$ or $m > 1$. For later use, we record the result for general ℓ. For ease of notation, for a power series $\alpha(z) \in \mathbb{C}[[z]]$ we shall write $\alpha(z) = \sum_j \alpha_j z^j$.

Lemma 8 *Let $X = \alpha(z)w^\ell\frac{\partial}{\partial z} + \beta(z)w^{\ell+1}$ be an infinitesimal CR automorphism of A, which we assume is given in normal form and satisfies* in $A = (1, m)$, *with $m > 1$. Then we can write*

$$\alpha(z) = -\bar{\alpha}_0\frac{B_{\chi^2}(z,0)}{2imz^{m-1}} + \frac{\ell\bar{\beta}_0 - \bar{\alpha}_1}{m}z + \bar{\alpha}_0\frac{2i(\ell+1)}{m}z^{m+1}, \qquad \beta(z) = \bar{\beta}_0 + 2i\bar{\alpha}_0 z^m. \tag{12}$$

Proof The tangency equation for X is

$$\beta(z)B(z,\chi)^{\ell+1} - \bar{\beta}(\chi)B(z,\chi) - \alpha(z)B(z,\chi)^\ell B_z(z,\chi) - \bar{\alpha}(\chi)B_\chi(z,\chi). \tag{13}$$

If we evaluate (13) along $\chi = 0$, the normalization conditions $B(z,0) = B(0,\chi) = 1$, $B_\chi(z,0) = 2iz^m$, $B_z(0,\chi) = 2i\chi^m$ imply

$$\beta(z) = \bar{\beta}_0 + 2i\bar{\alpha}_0 z^m. \tag{14}$$

In order to obtain an equation involving α, we take the derivative of (13) with respect to χ and evaluate along $\chi = 0$, substitute (14), and obtain (12). □

Lemma 9 *Let* $X = \alpha(z) w^\ell \frac{\partial}{\partial z} + \beta(z) w^{\ell+1}$ *be an infinitesimal CR automorphism of* A, *which we assume is given in normal form and satisfies* in $A = (1, 1)$. *Then*

$$\alpha(z) = \alpha_0 + (\ell\beta_0 - \bar\alpha_1) z + \bar\alpha_0 \big(2i\ell z^2 - A_{\chi^2}(z, 0)\big), \qquad \beta(z) = \bar\beta_0 + 2i\bar\alpha_0 z. \quad (15)$$

Proof We start with the same equation as in the proof of Lemma 7,

$$\beta(z) B(z, \chi)^{\ell+1} = \bar\beta(\chi) B(z, \chi) + \alpha(z) B_z(z, \chi) B(z, \chi)^\ell + \bar\alpha(\chi) B_\chi(z, \chi). \quad (16)$$

Evaluating for $\chi = 0$ gives

$$\beta(z) = \bar\beta(0) + \bar\alpha(0) B_\chi(z, 0) = \bar\beta(0) + \bar\alpha(0) 2iz; \quad (17)$$

in particular, $\beta(0) \in \mathbb{R}$. We next differentiate (16) with respect to χ, substitute (17), and evaluate for $\chi = 0$; solving the resulting equation for $\alpha(z)$ gives (15) after noting that $B_{\chi^2}(z, 0) = 2i A_{\chi^2}(z, 0) - 4z^2$. □

5 Exceptional Hypersurfaces

In the preceding section, we have seen that nonlinear automorphisms (for hypersurfaces given in normal form) are very restricted. Our goal in this section is to characterize the hypersurfaces which actually possess nonlinear symmetries; we will refer to them as “exceptional” hypersurfaces.

We first discuss the case of an infinitesimal CR automorphism as in Lemma 6 and start with the case $T = 0$.

Lemma 10 *Let* $2 \le k \in \mathbb{N}$, *and assume that*

$$p(z, \bar z) = \sum_{j=1}^{k-1} p_j z^j \bar z^{k-j}$$

is a real-valued polynomial (i.e., $p_j = \overline{p_{k-j}}$*). Then there exists a unique hypersurface of the form*

$$t = s\, A(z, \bar z),$$

where $A(z, \bar z) = p(z, \bar z) + O(|z|^{k+1})$ *with the property that the vector field*

$$X = \frac{zw}{k} \frac{\partial}{\partial z} + w^2 \frac{\partial}{\partial w}$$

is an infinitesimal CR automorphism of A, *or equivalently, which supports the* 1*-parameter group of automorphisms given for* $t \in \mathbb{R}$ *by*

$$z \mapsto \frac{z}{(1 - tw)^{\frac{1}{k}}}, \qquad w \mapsto \frac{w}{1 - tw}.$$

Proof We start with the existence proof. For this, we consider the following hyperquadric in $\mathbb{C}^{2k-1}$:

$$\operatorname{Im}\eta = \operatorname{Re}\left(\sum_{j=1}^{k-1} p_j \zeta_j \overline{\xi_j}\right). \tag{18}$$

We construct a hypersurface which supports the 1-parameter group of automorphisms obtained by transporting the 1-parameter group of automorphisms of the hyperquadric

$$(\zeta, \xi, \eta) \mapsto \frac{1}{1-t\eta}(\zeta, \xi, \eta)$$

via the blowup given by

$$\eta = w, \qquad \zeta_j = z^j w^{(1-\frac{j}{k})}, \qquad \xi_j = z^{k-j} w^{\frac{j}{k}}. \tag{19}$$

It is easy to check that this group gives rise to the group in the statement of the lemma, so we turn to proving that there is an (essentially unique, as the reader will see) real-analytic hypersurface in the preimage of the hyperquadric (18) by the blowup (19). For this purpose, it is more convenient to write our hypersurface in the equivalent form $w = \bar{w} e^{i\varphi(z,\bar{z})}$. The equation for φ is

$$e^{i\varphi} - 1 = i\left(\sum_j p_j z^j \bar{z}^{k-j} e^{i(1-\frac{j}{k})\varphi} + \sum_j \bar{p}_j \bar{z}^j z^{k-j} e^{i\frac{j}{k}\varphi}\right).$$

Clearly, this equation allows for a unique solution $\varphi(z,\bar{z})$ satisfying $\varphi(0,0) = 0$, and this solution has the property that $\varphi(z,\bar{z}) = 2p(z,\bar{z}) + O(|z|^{k+1})$. We only have to check that $\varphi(z,\bar{z})$ is real valued. Taking the complex conjugate of the equation defining φ, we see that $\bar{\varphi}(\bar{z}, z)$ satisfies the same equation as φ. By the uniqueness of the solution of this equation, we have that φ is real valued.

In order to prove the uniqueness part of the lemma, we consider any hypersurface of the form $w = \bar{w}\Theta(z,\bar{z})$ and assume that it allows the infinitesimal CR vector field given in the statement of the lemma. Θ thus satisfies the differential equation

$$\Theta(\Theta - 1) = \frac{z\Theta_z\Theta + \bar{z}\Theta_{\bar{z}}}{k}.$$

Substituting $\Theta = e^{i\varphi}$, we see that φ satisfies

$$\sin\left(\frac{1}{2}\varphi\right) = \frac{1}{k}\operatorname{Re}\left(z\varphi_z e^{i\frac{1}{2}\varphi}\right). \tag{20}$$

Again, one easily checks that this differential equation has for any given k-th degree real-valued polynomial p a uniquely determined solution φ with $\varphi(z,\bar{z}) = 2p(z,\bar{z}) + O(|z|^{k+1})$. This proves the uniqueness part. □

Lemma 11 *Let $\ell \in \mathbb{N}$ be positive, and $T \in \mathbb{R}$. Then there exists a unique hypersurface of the form*

$$t = sA(z, \bar{z})$$

with $A(z, \bar{z}) = \pm|z|^{2\ell} + O(|z|^{2\ell+1})$ which has the property that

$$X = \left(\frac{1}{2\ell} + iT\right) zw \frac{\partial}{\partial z} + w^2 \frac{\partial}{\partial w}$$

is an infinitesimal CR vector field of A, or equivalently, which supports the action of the 1-parameter group of automorphisms given for $t \in \mathbb{R}$ by

$$z \mapsto \frac{z}{(1 - tw)^{\frac{1}{2\ell} + iT}}, \qquad w \mapsto \frac{w}{1 - tw}.$$

Furthermore, A is of the form $A(z, \bar{z}) = A(|z|^{2\ell})$.

Proof Following the lead of the proof of Lemma 10 we now consider the hypersurface in $\mathbb{C}^2$ given by

$$\operatorname{Im} \eta = |\zeta|^{2\ell},$$

with its 1-parameter group of automorphisms given by

$$\zeta \mapsto \frac{\zeta}{(1 - t\eta)^{\frac{1}{\ell}}}, \qquad \eta \mapsto \frac{\eta}{1 - t\eta}.$$

We use the following blowup:

$$\zeta = zw^{(\frac{1}{2\ell} + iT)}, \qquad \eta = \pm w;$$

again, a simple computation shows that under this blowup, the 1-parameter group above gives rise to the 1-parameter group in the statement of the lemma. We thus prove existence of a real-analytic hypersurface in the preimage of the hypersurface above under these blowups.

We write the defining equation (as before) in the form $w = \bar{w}\, e^{i\varphi(z,\bar{z})}$. If a hypersurface of this form is in the preimage of the hypersurface above under the blowup, then we have the following equation for φ:

$$\sin\left(\frac{1}{2}\varphi\right) = \pm|z|^{2\ell}\, e^{-\ell T \varphi}. \tag{21}$$

Obviously this equation has a unique solution, which furthermore satisfies $\varphi(z, \bar{z}) = \varphi(|z|^{2\ell}) = \pm 2|z|^{2\ell} + O(|z|^{2\ell+1})$.

Similarly as in the proof of Lemma 10 we can also give a characterization by a differential equation. In this case, the equation after a little computation turns out to be

$$t\varphi'(t) = \frac{2\sin(\frac{1}{2}\varphi(t))}{\frac{1}{\ell}\cos(\frac{1}{2}\varphi(t)) + \frac{T}{\ell}\sin(\frac{1}{2}\varphi(t))}. \tag{22}$$

□

To finish our presentation of the exceptional hypersurfaces, we still need to treat the case of an infinitesimal symmetry as in Lemmas 8 or 9, i.e., we assume that we have an infinitesimal automorphism of the form $\alpha(z)w\frac{\partial}{\partial z}+\beta(z)w^2\frac{\partial}{\partial w}$ with $\alpha(0)\neq 0$. However, differently from the cases considered before, these families of hypersurfaces are infinite dimensional; we shall discuss them in more detail in Sect. 6.

Lemma 12 *Let $\theta, s, t \in \mathbb{R}$ be parameters, $\varphi(t)=\sum_{j=2}^{\infty} a_j t^j$ be a real power series (i.e., $a_j=\bar{a}_j$ for all j) and $a_2=\pm 1$. Then there exists a unique hypersurface of the form*

$$t = s\,A(z,\bar{z})$$

with A in normal form, in $A=(1,1)$, *and* it $A=a_2$, *such that with $A(z,\chi)=\sum_{j,k} A_{j,k} z^j \chi^k$ we have $A_{j,j}=a_{2j}$ and* $\operatorname{Im} e^{i\theta} A_{j,j-1}=a_{2j-1}$ *and such that A has an infinitesimal automorphism of the form*

$$X=\left(e^{i\theta}+z(s+it)+e^{-i\theta}\left(2iz^2-A_{\chi^2}(z,0)\right)\right)w\frac{\partial}{\partial z}+\left(2s+2ie^{-i\theta}z\right)w^2\frac{\partial}{\partial w}.$$

Proof We already know by Lemma 9 that if we have a hypersurface A with in $A=(1,1)$ which possesses an infinitesimal automorphism of the form $X=\alpha(z)w\frac{\partial}{\partial z}+\beta(z)w^2\frac{\partial}{\partial w}$ with $\alpha(0)\neq 0$, then after possibly rescaling, X is of the form stated above. Such an X is tangent to A if and only if

$$\beta(z)\left(1+iA(z,\chi)\right)-\bar{\beta}(\chi)\left(1-iA(z,\chi)\right)-2i\left(\frac{\alpha(z)A_z(z,\chi)}{1-iA(z,\chi)}+\frac{\bar{\alpha}(\chi)A_\chi(z,\chi)}{1+iA(z,\chi)}\right)=0.$$

Writing $A(z,\chi)=\sum_j A_j(z,\chi)$ with A_j homogeneous of degree j, and similarly $\beta(z)=\sum_j \beta_j z^j$ and $\alpha(z)=\sum_j \alpha_j z^j$, we see that the A_j are determined recursively by

$$\begin{aligned}\alpha_0 A_{j,z}(z,\chi)+\bar{\alpha}_0 A_{j,\chi}(z,\chi)+\alpha_{j-2}z^{j-2}\chi+\bar{\alpha}_{j-2}z\chi^{j-2}&\\ +\frac{1}{2}\left(\beta_{j-1}z^{j-1}+\bar{\beta}_{j-1}\chi^{j-1}\right)&=p(z,\chi),\end{aligned}$$

where $p(z,\chi)$ is a polynomial in z,χ of degree $j-1$ with coefficients which are polynomials in the coefficients of A_k for $k<j-1$, α_k and $\bar{\alpha}_k$ for $k<j-2$, β_k and $\bar{\beta}_k$ for $k<j-1$, and satisfies $p_{j-1}(z,\chi)=\bar{p}_{j-1}(\chi,z)$. Writing out the equations for the coefficients $A_{j-\ell,\ell}$,

$$\begin{aligned}\alpha_0(j-\ell)A_{j-\ell,\ell}+\bar{\alpha}_0(\ell+1)A_{j-\ell-1,\ell+1}&=p_{j-\ell-1,\ell}, \quad 2\le \ell<j-2\\ \alpha_0(j-1)A_{j-1,1}+\bar{\alpha}_0 2A_{j-2,2}+\alpha_{j-2}&=p_{j-2,1},\\ \alpha_0 jA_{j,0}+\bar{\alpha}_0 A_{j-1,1}+\frac{1}{2}\beta_{j-1}&=p_{j-1,0},\\ \alpha_0 2A_{2,j-2}+\bar{\alpha}_0(j-1)A_{1,j-1}+\bar{\alpha}_{j-2}&=p_{1,j-2},\\ \alpha_0 A_{1,j-1}+\bar{\alpha}_0 jA_{0,j}+\frac{1}{2}\bar{\beta}_{j-1}&=p_{0,j-1},\end{aligned} \tag{23}$$

we see that this set of equations has a solution $(A_{j-\ell,\ell})_{\ell=0,\dots,j}$ satisfying $A_{j-\ell,\ell} = \bar{A}_{\ell,j-\ell}$ which is uniquely determined by one real parameter: for even $j = 2k$ the real number $A_{k,k}$ and for odd $j = 2k-1$ the parameter $\operatorname{Im} k\alpha_0 A_{k,k-1}$; if we furthermore require that $A_{j-1,0} = A_{j-2,1} = 0$ (for $j \geq 2$), α_{j-1} and β_{j-1} are uniquely determined, and the resulting A is in normal form. □

Lemma 13 *Let $\varphi(t) = \sum_{j=m}^{\infty} a_j t^j$ a real power series (i.e., $a_j = \bar{a}_j$ for all j) and $a_m = 1$. Then there exists a unique hypersurface of the form*

$$t = s\,A(z, \bar{z})$$

with A in normal form, $\operatorname{in} A = (1, m)$, *and* $\operatorname{it} A = 1$, *such that with $A(z,\chi) = \sum_{j,k} A_{j,k} z^j \chi^k$ we have $A_{j,j} = a_{2j}$ and* $\operatorname{Re} A_{j,j-1} = a_{2j-1}$ *and such that A has an infinitesimal automorphism of the form*

$$X = \left(\frac{i A_{\chi^2}(z,0)}{m z^{m-1}} + z(\beta_0 - \bar{\alpha}_1) + 2z^{m+1} \right) w \frac{\partial}{\partial z} + \left(\beta_0 + 2z^m\right) w^2 \frac{\partial}{\partial w}.$$

Proof We already know by Lemma 8 that if we have a hypersurface A with $\operatorname{in} A = (1, m)$ which possesses an infinitesimal automorphism of the form $X = \alpha(z) w \frac{\partial}{\partial z} + \beta(z) w^2 \frac{\partial}{\partial w}$ with $\alpha(0) \neq 0$, then after possibly rescaling and a rotation, X is of the form stated above. Such an X is tangent to A if and only if

$$\beta(z)\big(1+iA(z,\chi)\big) - \bar{\beta}(\chi)\big(1-iA(z,\chi)\big) - 2i\left(\frac{\alpha(z) A_z(z,\chi)}{1-iA(z,\chi)} + \frac{\bar{\alpha}(\chi) A_\chi(z,\chi)}{1+iA(z,\chi)} \right) = 0.$$

We now proceed as in the proof of Lemma 12 to construct the defining equation. □

6 Tube-Like Realizations

We have already encountered an infinite-dimensional family of hypersurfaces supporting a family of nonlinear automorphisms in normal form in Lemmas 12 and 13; however, the construction in these lemmas is not particularly illuminating with respect to the nature of these hypersurfaces. In this section, we shall present another realization of these hypersurfaces which exhibits them as tube-like hypersurfaces, and give a corresponding tube-like normalization. First note that it is enough to consider the cases where we are dealing with a CR automorphism of the form

$$\alpha(z) w^\ell \frac{\partial}{\partial z} + \beta(z) w^{\ell+1} \frac{\partial}{\partial w},$$

with $\alpha(0) \neq 0$, where $\ell = 0$ or $\ell = 1$; the general case is recovered by an application of Lemma 5. Even though the proof of the next lemma is given by straightening a nonsingular vector field, we will need the special form of change of coordinates later.

Proposition 14 *Let M be a ruled hypersurface which supports an infinitesimal CR automorphism of the form $X = \alpha(z)\frac{\partial}{\partial z} + \beta(z) w \frac{\partial}{\partial w}$ with $\alpha(0) \neq 0$. Then there exist coordinates $(\zeta, \xi) \in \mathbb{C}^2$ such that in these coordinates, M is given by* $\operatorname{Im} \xi =$

$(\operatorname{Re}\xi) f(\operatorname{Re}\zeta)$. *In these coordinates, X generates the 1-parameter group of translations $\zeta \mapsto \zeta + ia$ for $a \in \mathbb{R}$.*

Proof We introduce new coordinates $(\tilde{\zeta}, \xi)$ by setting $\tilde{\zeta} = z$ and $\xi = C(z)w$. In these coordinates, X becomes

$$X = \alpha(\tilde{\zeta}) \frac{\partial}{\partial \tilde{\zeta}} + \big(\beta(\tilde{\zeta}) C(\tilde{\zeta}) + C'(\tilde{\zeta}) \alpha(\tilde{\zeta})\big) \frac{\xi}{C(\tilde{\zeta})} \frac{\partial}{\partial \xi}.$$

Thus, we need to solve the ODE $\frac{C'(\tilde{\zeta})}{C(\tilde{\zeta})} = -\frac{\beta(\tilde{\zeta})}{\alpha(\tilde{\zeta})}$ in order to make $X = \alpha(\tilde{\zeta}) \frac{\partial}{\partial \tilde{\zeta}}$. We next straighten the vector field X to have $X = i \frac{\partial}{\partial \zeta}$, from which the claims of the lemma easily follow. □

Essentially the same proof as above applies in the case $\ell = 1$, so we summarize the result:

Proposition 15 *Let M be a ruled hypersurface which supports an infinitesimal CR automorphism of the form $X = \alpha(z) w \frac{\partial}{\partial z} + \beta(z) w^2 \frac{\partial}{\partial w}$ with $\alpha(0) \neq 0$. Then there exist coordinates $(\zeta, \xi) \in \mathbb{C}^2$ such that in these coordinates, M is given by $\operatorname{Im}\xi = (\operatorname{Re}\xi) f(\operatorname{Re}\zeta, \operatorname{Im}\zeta)$ and $f(x, y)$ is a solution of the PDE $f_y - f f_x = 0$, or equivalently, $\xi = \bar{\xi} g(\zeta, \bar{\zeta})$ where $g g_\zeta - g_{\bar{\zeta}} = 0$. In these coordinates, X generates the 1-parameter group $(\zeta, \xi) \mapsto (\zeta + ia\xi, \xi)$ for $a \in \mathbb{R}$.*

Remark 3 The PDE satisfied by the defining function $f(x, y)$ in Proposition 15,

$$\frac{\partial f}{\partial y} - f \frac{\partial f}{\partial x} = 0,$$

is the *transport equation* which one can solve from Cauchy data $u(x) = f(x, 0)$ by solving the equation

$$u\big(x + f(x, y) y\big) = f(x, y).$$

One can actually derive this fact by an application of Proposition 15, since the flow of the 1-parameter group given there dictates that

$$f(x, y) = f\big(x - s f(x, y), y + s\big).$$

Using $s = -y$ to restrict to suitable Cauchy data $f(x, 0)$ gives the equation above.

In particular, we see that for an arbitrary ruled hypersurface allowing an infinitesimal automorphism $X = \alpha(z) w^\ell \frac{\partial}{\partial z} + \beta(z) w^{\ell+1} \frac{\partial}{\partial w}$ with $\ell > 0$, we can choose coordinates (ζ, ξ) such that the 1-parameter group generated by X in these coordinates is given by $\zeta \mapsto \zeta + ia\xi^\ell$; the defining function in these coordinates is $\xi = \bar{\xi} g(\zeta, \bar{\zeta})$ where $g^{1/\ell}$ satisfies the PDE from Proposition 15.

By Proposition 14 we can identify the space $\mathcal{T}_0$ of ruled hypersurfaces which allow an automorphism of the form $\alpha(z) \frac{\partial}{\partial z} + \beta(z) w \frac{\partial}{\partial w}$ with $\alpha(0) \neq 0$ with

$$\mathcal{T}_0 = \big\{ B(z, \chi) \colon B(z, \chi) \bar{B}(\chi, z) = 1, B_z - B_\chi = 0 \big\}$$

$$= \left\{A(x,y)\colon \left(\mathbb{R}^2,0\right)\to(\mathbb{R},0),\, A(x,y)=A(x,0)\right\};$$

and by Proposition 15, the space $\mathcal{T}_1$ which allow an automorphism of the form $\alpha(z)w\frac{\partial}{\partial z}+\beta(z)w^2\frac{\partial}{\partial w}$ with $\alpha(0)\neq 0$ with

$$\begin{aligned}\mathcal{T}_1 &= \left\{B(z,\chi)\colon B(z,\chi)\bar{B}(\chi,z)=1,\, B_\chi - BB_z=0\right\}\\ &= \left\{A(x,y)\colon \left(\mathbb{R}^2,0\right)\to(\mathbb{R},0),\, A_y-AA_x=0\right\}.\end{aligned}$$

Even more generally, we could also consider the space

$$\mathcal{T}_\ell = \left\{B(z,\chi)\colon B(z,\chi)\bar{B}(\chi,z)=1,\, B_\chi - B^\ell B_z=0\right\}.$$

However, an easy computation along the lines of Lemma 5 shows that $B\in\mathcal{T}_\ell$ if and only if $B^\ell\in\mathcal{T}_1$.

We can therefore treat the biholomorphic equivalence problem for hypersurfaces which allow an infinitesimal automorphism with $\alpha(0)\neq 0$ as a normalization in $\mathcal{T}_0$ or $\mathcal{T}_1$, respectively. In this section, we give an explicit solution to this problem in Theorems 6 and 7, respectively, which also describe the moduli space of hypersurfaces under biholomorphic equivalence in $\mathcal{T}_0$ and $\mathcal{T}_1$ in terms of special normalized Cauchy data.

We start with the case of $\ell=0$ and first extract a basic invariant. Let $H=(f,g)$ be a biholomorphism with $H(0)=0$ taking two hypersurfaces in $\mathcal{T}_0$, defined by B and $\tilde{B}$, respectively, into each other, i.e.,

$$g\big(z,\tau B(z,\chi)\big)=\bar{g}(\chi,\tau)\tilde{B}\big(f\big(z,\tau B(z,\chi)\big),\bar{f}(\chi,\tau)\big).$$

Comparing the coefficients of τ on both sides of the equation gives $g_w(z,0)B(z,\chi)=\bar{g}_\tau(\chi,0)\tilde{B}(f(z,0),\bar{f}(\chi,0))$, i.e.,

$$B(z,\chi)=\frac{\bar{g}_\tau(\chi,0)}{g_w(z,0)}\tilde{B}\big(f(z,0),\bar{f}(\chi,0)\big).$$

If we take a derivative with respect to χ and set $\chi=0$, we obtain

$$\begin{aligned}B_\chi(z,0) &= \frac{\bar{g}_{\chi\tau}(0)}{g_w(z,0)}\tilde{B}\big(f(z,0),0\big)+\frac{\bar{g}_\tau(0)}{g_w(z,0)}\tilde{B}_\chi\big(f(z,0),0\big)\bar{f}_\chi(0)\\ &= \frac{\bar{g}_{\chi\tau}(0)}{\bar{g}_\tau(0)}B(z,0)+\frac{B(z,0)}{\tilde{B}(f(z,0),0)}\tilde{B}_\chi\big(f(z,0),0\big)\bar{f}_\chi(0)\\ &= \big(B_\chi(0,0)-\tilde{B}_\chi(0,0)\bar{f}_\chi(0)\big)B(z,0)\\ &\quad+\frac{B(z,0)}{\tilde{B}(f(z,0),0)}\tilde{B}_\chi\big(f(z,0),0\big)\bar{f}_\chi(0),\end{aligned}$$

and hence

$$\frac{B_\chi(z,0)}{B(z,0)}-B_\chi(0,0)=\bar{f}_\chi(0)\left(\frac{\tilde{B}_\chi(f(z,0),0)}{\tilde{B}(f(z,0),0)}-\tilde{B}_\chi(0,0)\right).$$

Thus, for $B, \tilde{B} \in \mathcal{T}_0$, the logarithmic derivative of $B(z,0) = b(z)$ transforms as follows under a biholomorphism:

$$\frac{b'(z)}{b(z)} - b'(0) = \bar{f}_\chi(0)\left(\frac{\tilde{b}'(f(z,0))}{\tilde{b}(f(z,0))} - \tilde{b}'(0)\right). \tag{24}$$

In particular, the vanishing order $\nu_0(B)$ of $\psi(z) = \frac{b'(z)}{b(z)} - b'(0)$ is a biholomorphic invariant of the hypersurface described by B. We will in this section only treat the case of hypersurfaces where $\nu_0(B) \geq 2$; the case $\nu_0(B) = 1$ will be treated later. If B_d is the distinguished normal form of B, this is equivalent to in $B_d = (1, m)$ with $m \geq 2$ (a straightforward calculation shows that $\nu_0(B) = m$). Let us write $\mathcal{T}_0^\nu$ for the space of hypersurfaces in $\mathcal{T}_0$ with $\nu_0(B) = \nu$.

Lemma 16 *Assume that B is in distinguished normal form, with* in $B = (1, m)$ *and $m > 1$. If $X = \alpha(z)\frac{\partial}{\partial z} + \beta(z)w\frac{\partial}{\partial w}$ and $Y = \gamma(z)\frac{\partial}{\partial z} + \delta(z)w\frac{\partial}{\partial w}$ are infinitesimal CR automorphisms of B with $\alpha(0) \neq 0$, $\gamma(0) \neq 0$, then $X = tY$ for some $t \in \mathbb{R}$ modulo $w\frac{\partial}{\partial w}$.*

Proof From Lemma 8, we have

$$\alpha(z) = -\bar{\alpha}_0 \frac{B_{\chi^2}(z,0)}{2imz^{m-1}} - \frac{\bar{\alpha}_1}{m}z + \bar{\alpha}_0\frac{2i}{m}z^{m+1}. \tag{25}$$

Similar equations hold for Y. We note that this implies that, if $\alpha_0 \neq 0$, $\arg \alpha_0$ is uniquely determined by the coefficient of z^{m-1} in $B_{\chi^2}(z,0)$; we thus have that $\alpha_0 = t\gamma_0$ for some $t \neq 0$.

Now (25) implies that $X - tY = sz\frac{\partial}{\partial z}$ modulo $w\frac{\partial}{\partial w}$ for some $s \in \mathbb{C}$, which, again by (25) satisfies $ms + \bar{s} = 0$. So we have $s = 0$ and $X = tY$ modulo $w\frac{\partial}{\partial w}$ as required. □

Proposition 17 *Two ruled hypersurfaces $B(z,\chi) = b(z+\chi)$, $\tilde{B}(z,\chi) = \tilde{b}(z+\chi)$ in $\mathcal{T}_0^\nu$, where $\nu \geq 2$, are biholomorphically equivalent if and only if there exist $s, t \in \mathbb{R}$ such that $\tilde{b}(z) = e^{itz}b(sz)$.*

Proof If B and $\tilde{B}$ are biholomorphically equivalent via $H(z,w) = (f(z,w), g(z,w))$, then they are also biholomorphically equivalent via $H_0(z,w) = (f(z,0), g_w(z,0)w)$; we thus assume that we are given a biholomorphism of the form $H(z,w) = (f(z), g(z)w)$. Now Lemma 16 implies that there are only 2 linearly independent infinitesimal automorphisms of homogeneity 0 tangent to B and the same for $\tilde{B}$, namely, $i\frac{\partial}{\partial z}$ and $w\frac{\partial}{\partial w}$. We compute the push of $i\frac{\partial}{\partial z}$ via H, which is given by

$$\frac{i}{f'(z)}\frac{\partial}{\partial z} - \frac{ig'(z)}{g(z)f'(z)}w\frac{\partial}{\partial w}.$$

Since this has to be an $\mathbb{R}$-linear combination of $i\frac{\partial}{\partial z}$ and $w\frac{\partial}{\partial w}$, we find an $s\in\mathbb{R}$, $s\neq 0$ with $s=f'(z)$ and with some $t\in\mathbb{R}$,

$$i\frac{g'(z)}{g(z)}=t,$$

i.e., $g(z)=e^{itz}$. This implies that $\tilde{b}(z)=e^{itz}b(sz)$. If on the other hand this equation holds, then the biholomorphism $(sz,e^{itz}w)$ takes B to $\tilde{B}$. □

We can thus normalize the hypersurfaces in $\mathcal{T}_0^\nu$ in the following way:

Theorem 6 *Every hypersurface in $\mathcal{T}_0^\nu$, where $\nu\geq 2$, is biholomorphically equivalent to a unique hypersurface in $\mathcal{N}_0^\nu$, where*

$$\mathcal{N}_0^\nu=\left\{B(z,\chi)=b(z+\chi)\in\mathcal{T}_0^\nu : b'(0)=0, b^{(\nu+1)}(0)=i(\nu+1)!\right\}.$$

In particular, the modulus space of $\mathcal{T}_0^\nu$ under biholomorphic equivalence can be identified with the space of all power series of the form $x^{\nu+1}+\sum_{j\geq\nu+2}a_jx^j$, $a_j\in\mathbb{R}$.

For $\mathcal{T}_1$, we obtain a similar invariant ν_1 as above; however, in this case, (24) becomes replaced by (we again write $B(z,0)=b(z)$)

$$b'(z)-b'(0)=\bar{f}_\chi(0)\bigl(\tilde{b}'\bigl(f(z,0)\bigr)-\tilde{b}'(0)\bigr). \tag{26}$$

We thus define $\nu_1(B)=\operatorname{ord}(B_z(z,0)-B_z(0,0))$ for $B\in\mathcal{T}_1$, or more generally $\nu_\ell(B)=\operatorname{ord}(B_z(z,0)-B_z(0,0))$ for $B\in\mathcal{T}_\ell$. The spaces $\mathcal{T}_\ell^\nu$ are defined as above.

Lemma 18 *Assume that B is in distinguished normal form, with* in $B=(1,m)$, *and $m>1$. Then, if $X=\alpha(z)w\frac{\partial}{\partial z}+\beta(z)w^2\frac{\partial}{\partial w}$ and $Y=\gamma(z)w\frac{\partial}{\partial z}+\delta(z)w^2\frac{\partial}{\partial w}$ are infinitesimal CR automorphisms of B with $\alpha(0)\neq 0$, $\gamma(0)\neq 0$, then $X=tY$ for some $t\in\mathbb{R}$.*

Proof From Lemma 8, we have

$$\alpha(z)=-\bar{\alpha}_0\frac{B_{\chi^2}(z,0)}{2imz^{m-1}}+\frac{\bar{\beta}_0-\bar{\alpha}_1}{m}z+\bar{\alpha}_0\frac{4i}{m}z^{m+1},\qquad \beta(z)=\bar{\beta}_0+2i\bar{\alpha}_0z^m,$$

with similar equations for γ and δ. We notice that this implies that for some $t\in\mathbb{R}$, $\alpha_0=t\gamma_0$. Another application of Lemma 8 then shows that $Z=X-tY$ is an infinitesimal automorphism of B of the form

$$Z=\frac{rz}{m+1}w\frac{\partial}{\partial z}+rw^2\frac{\partial}{\partial w}.$$

An easy computation shows that $(\operatorname{ad}Z)^\ell X$, if $r\neq 0$, is then a nontrivial infinitesimal automorphism and

$$(\operatorname{ad}Z)^\ell X=a_\ell(z)w^{\ell+1}\frac{\partial}{\partial z}+b_\ell(z)w^{\ell+2}\frac{\partial}{\partial w}.$$

Since the space of infinitesimal automorphisms of B is finite dimensional, it follows that $r = 0$, and $X = tY$ as required. □

Proposition 19 *Two ruled hypersurfaces* $A(x, y)$ *and* $\tilde{A}(x, y)$ *in* $\mathcal{T}_1^\nu$, *where* $\nu \geq 2$, *are biholomorphically equivalent if and only if there exists a* $t \in \mathbb{R}$, $t \neq 0$, *such that* $A(x, 0) = \tilde{A}(tx, 0)$.

Proof As in the proof of Proposition 17, we see that if $H(z, w) = (f(z, w), g(z, w))$ is a biholomorphism between A and $\tilde{A}$, then $H_0(z, w) = (f(z, 0), wg_w(z, 0)) =: (f(z), g(z)w)$ is one, as well. From Lemma 18 we see that the dimension of $\mathfrak{g}(A)_1$, as well as the one of $\mathfrak{g}(\tilde{A})_1$, is 1. Since $iw\frac{\partial}{\partial z} \in \mathfrak{g}(A)_1$ and $iw\frac{\partial}{\partial z} \in \mathfrak{g}(\tilde{A})_1$ by the definition of $\mathcal{T}_1$, we have $(H_0)_* iw\frac{\partial}{\partial z} = siw\frac{\partial}{\partial z}$ for some $s \neq 0$.

We calculate that

$$\frac{i}{f'(z)}\left(wg(z)\frac{\partial}{\partial z} - g'(z)w^2\frac{\partial}{\partial w}\right) = isw\frac{\partial}{\partial z},$$

from which it follows that $g'(z) = 0$, i.e., $g(z) = g(0)$, and $f'(z) = 1/s$, i.e., $f(z) = \frac{z}{s}$. This proves the statement in the proposition. □

Theorem 7 *Every hypersurface in* $\mathcal{T}_1^\nu$, *where* $\nu \geq 2$, *is biholomorphically equivalent to a unique hypersurface in* $\mathcal{N}_1^\nu$, *where*

$$\mathcal{N}_1^\nu = \left\{A(x, y) \in \mathcal{T}_1^\nu : A_{x^{\nu+1}}(0, 0) = (\nu + 1)!\right\}.$$

In particular, the moduli space of $\mathcal{T}_1^\nu$ *under biholomorphic equivalence can be identified with the space of all power series of the form* $x^{\nu+1} + \sum_{j \geq \nu+2} a_j x^j$, $a_j \in \mathbb{R}$.

7 Degenerate Tubes

In this section we will discuss the infinitesimal automorphisms of a ruled hypersurface which is a degenerate tube: by this, we mean hypersurfaces which allow an infinitesimal automorphism of the form $\alpha(z)w^\ell\frac{\partial}{\partial z} + \beta(z)w^{\ell+1}\frac{\partial}{\partial w}$ with $\alpha(0) \neq 0$ whose defining equation B satisfies $\nu_\ell(B) \geq 2$. Our starting point is the following observation.

Proposition 20 *Assume that* $B(z, \chi)$ *is a ruled hypersurface, given in distinguished normal form, with* in $B = (1, m)$ *where* $m \geq 2$. *If* $X = \alpha(z)w^\ell\frac{\partial}{\partial z} + \beta(z)w^{\ell+1}\frac{\partial}{\partial w}$ *and* $Y = \gamma(z)w^k\frac{\partial}{\partial z} + \delta(z)w^{k+1}\frac{\partial}{\partial w}$ *are infinitesimal automorphisms of* B *satisfying* $\alpha(0) \neq 0$, $\gamma(0) \neq 0$, $[X, Y] = 0$, *and* $k \geq \ell$, *then* $k = (m + 1)\ell$. *If* $k = \ell \neq 0$, *then* $X = tY$ *for some* $t \in \mathbb{R}$.

Proof From $[X, Y] = 0$ we have that

$$\alpha\gamma' - \gamma\alpha' + k\beta\gamma - \ell\delta\alpha = 0, \qquad \alpha\delta' - \gamma\beta' + (k - \ell)\beta\delta = 0. \tag{27}$$

From Lemma 8 we know that after rescaling Y, we have that $\alpha_0 = \gamma_0$; the second equation from (27) now immediately implies the statement for $k = \ell$. A further application of Lemma 8 shows that α_1 and γ_1 satisfy

$$m\alpha_1 + \bar{\alpha}_1 = A + \ell\beta_0, \qquad m\gamma_1 + \bar{\gamma}_1 = A + k\delta_0,$$

i.e., $\alpha_1 - \gamma_1 = \frac{\ell\beta_0 - k\delta_0}{m+1}$. Similarly, we have that $\alpha_{m+1} - \gamma_{m+1} = \frac{2i\bar{\alpha}_0(\ell-k)}{m}$. Another application of Lemma 8 shows that $\alpha = \gamma + (\alpha_1 - \gamma_1)z + (\alpha_{m+1} - \gamma_{m+1})z^{m+1}$. We can thus rewrite the second equation from $[X, Y] = 0$ from (27) as

$$\begin{aligned} 0 &= \alpha(z)\delta'(z) - \gamma(z)\beta'(z) + (k-\ell)\beta(z)\delta(z) \\ &= 2i\bar{\alpha}_0 m z^{m-1}\big(\alpha(z) - \gamma(z)\big) + (k-\ell)\big(\beta_0\delta_0 + 2i\bar{\alpha}_0(\beta_0+\delta_0)z^m - 4\bar{\alpha}_0^2 z^{2m}\big). \quad (28) \end{aligned}$$

We conclude from comparing the coefficients of z, z^m, and z^{2m} in the equation above that

$$\begin{aligned} (k-\ell)\beta_0\delta_0 &= 0, \\ \beta_0\big(\ell - (m+1)k\big) - \delta_0\big(k - (m+1)\ell\big) &= 0. \end{aligned} \qquad (29)$$

If we assume that $\beta_0 = \delta_0 = 0$, we have that $\alpha_1 = \gamma_1$ and consequently that $\alpha = \gamma + \frac{2i\bar{\alpha}_0(\ell-k)}{m} z^{m+1} = \gamma + cz^{m+1}$. Using the first equation from (27), we see that

$$cz\gamma'(z) - c\gamma(z) - 2i\bar{\alpha}_0\ell c z^{2m+1} = 0.$$

Since this equation has a solution with $\gamma(0) \neq 0$, we have $c = 0$ and thus $k = \ell$.

We are left over with the possibility that exactly one of β_0 or δ_0 is not zero (if they are both nonzero, then $k = \ell$ from the first equation in (29)). However, since we assume that $k \geq \ell$, we have that $\ell - (m+1)k \neq 0$ (or, $k = \ell = 0$). Hence necessarily $\delta_0 \neq 0$ and $k = (m+1)\ell$ from the second equation in (29). □

We start with examining the case of degenerate tubes in $\mathcal{T}_0$.

Theorem 8 *Assume that* $B \in \mathcal{T}_0^\nu$, *with* $\nu \geq 2$. *Then* $\dim \mathfrak{hol}(B) = 2$.

Proof Assume that $X = \alpha(z)w^\ell \frac{\partial}{\partial z} + \beta(z)w^{\ell+1}\frac{\partial}{\partial w}$ is an infinitesimal automorphism of B. Proposition 20 shows that $\dim \mathfrak{g}(B)_\ell \leq 1$. Using the fact that $[i\frac{\partial}{\partial z}, X] = sX$ for some $s \in \mathbb{R}$, we see that

$$i\alpha'(z) = s\alpha(z), \qquad i\beta'(z) = s\beta(z).$$

We thus have $\alpha(z) = \alpha_0 e^{-isz}$ and $\beta(z) = \beta_0 e^{-isz}$. Since X is an infinitesimal automorphism, we have that

$$\beta(z)B(z,\chi)^{\ell+1} - \bar{\beta}(\chi)B(z,\chi) - \alpha(z)B(z,\chi)^\ell B_z(z,\chi) - \bar{\alpha}(\chi)B_\chi(z,\chi) = 0.$$

If $\alpha_0 \neq 0$, we find that if we put B into distinguished normal form $\tilde{B}$, then the corresponding infinitesimal automorphism $\tilde{X}$ is of the form

$$\tilde{X} = \tilde{\alpha}(z)w^\ell\frac{\partial}{\partial z} + \tilde{\beta}(z)w^{\ell+1}\frac{\partial}{\partial w},$$

with $\tilde{\alpha}(0) \neq 0$. But by assumption, there exists a nontrivial infinitesimal automorphism Y of homogeneity 0 modulo $w\frac{\partial}{\partial w}$. Hence, $[Y, \tilde{X}] = t\tilde{X}$ for some $t \in \mathbb{R}$, and thus $[Y - tw\frac{\partial}{\partial w}, \tilde{X}] = 0$. Now Proposition 20 implies that $\ell = 0$. This in turn shows that $\tilde{X} = Y$ modulo $w\frac{\partial}{\partial w}$.

If on the other hand $\alpha_0 = 0$, we have that $B(z, \chi) = b(z + \chi)$ is given by

$$b(x) = \frac{\beta_0}{\bar{\beta}_0} e^{-isx},$$

for which the vanishing order is $\nu(b) = \infty$. Indeed, the hypersurface defined by this equation is given by $w = \bar{w} e^{-is(z+\chi)}$, which is equivalent to $w = \bar{w}$ and thus Levi-flat. □

Theorem 9 *Assume that B has an infinitesimal automorphism of the form $w^\ell \frac{\partial}{\partial z}$ for some $\ell > 0$ and that $\nu(B) \geq 2$. Then* $\dim \mathfrak{hol}(B) = 2$.

Proof We start by noting that Lemma 16 implies that $\dim \mathfrak{g}(B)_0 = 1$. We choose ℓ in such a way that it is the maximum ℓ such that one can choose coordinates in which $w^\ell \frac{\partial}{\partial z}$ is an infinitesimal automorphism of B. Now Proposition 20 shows that we can write $\mathfrak{g}(B) = \mathfrak{g}(B)_0 \oplus \mathfrak{g}(B)_{\frac{\ell}{m+1}} \oplus \mathfrak{g}(B)_\ell$, all other homogeneous parts being 0. An application of Lemma 5 shows that by passing to $B^{1/\ell}$ (which we are again denoting by B), we can assume that $\ell = m + 1$, and that the decomposition is given by $\mathfrak{g}(B) = \mathfrak{g}(B)_0 \oplus \mathfrak{g}(B)_1 \oplus \mathfrak{g}(B)_{m+1}$.

We thus have the infinitesimal automorphisms $X = iw^{m+1} \frac{\partial}{\partial z}$ and $Y = \alpha(z) w \frac{\partial}{\partial z} + \beta(z) w^2 \frac{\partial}{\partial w}$. We only need to show that $Y = 0$. Since $[X, Y] = 0$, we have that

$$\alpha'(z) - (m+1)\beta(z) = 0, \qquad \beta'(z) = 0,$$

and thus

$$\beta(z) = \beta_0, \qquad \alpha(z) = \alpha_0 + (m+1)\beta_0 z.$$

We also know that

$$\beta_0 B(z, \chi)^2 - \bar{\beta}_0 B(z, \chi) - \alpha(z) B(z, \chi) B_z(z, \chi) - \bar{\alpha}(\chi) B_\chi(z, \chi) = 0.$$

Since $\nu(B) \geq 2$, we notice that $\beta_0 = \bar{\beta}_0$, and after rescaling we can assume that $\alpha_0 = i$. After substituting $B_z = B^{m+1} B_\chi$, we thus arrive at the following ODE for $b(z) = B(z, 0)$:

$$\beta_0 b(z) - \bar{\beta}_0 - \big(\alpha(z) + \bar{\alpha}_0 b(z)^m\big) b'(z) = 0.$$

Let us write $b(z) = 1 + z^k c(z)$ with $c(0) = c_0 \neq 0$. The preceding equation now reads in terms of $c(z)$ as

$$\beta_0 z^k c(z) - \left((m+1)\beta_0 z - i \sum_{j=1}^{m} \binom{m}{j} z^{jk} c(z)^j \right) \big(z^k c'(z) + k z^{k-1} c(z)\big) = 0.$$

Comparing the coefficients of z^k we see that, since $k \geq 2$ by assumption, necessarily $\beta_0 = 0$. This in turn implies that $b(z) = 1$ if $\alpha \neq 0$; hence $Y = 0$ and our claim is proved. □

8 Nondegenerate Tubes: Dimensions of $\mathfrak{g}_0$

In this section, we will start analyzing the infinitesimal automorphisms of a *nondegenerate tube*, i.e., the case $\nu(B) = 1$. We will discuss first the infinitesimal automorphisms in $\mathfrak{g}_0$. We put $(M, 0)$ into normal form, that is, we assume that the defining equation of $(M, 0)$ is given by

$$t = sA(z, \bar{z}), \qquad A(z, 0) = A(0, \bar{z}) = 0, \qquad A_z(0, \bar{z}) = i\bar{z}, \qquad A_{\bar{z}}(z, 0) = iz.$$

We know that $w\frac{\partial}{\partial w} \in \mathfrak{g}_0$. If a vector field $X = \alpha(z)\frac{\partial}{\partial z} + \beta(z)w\frac{\partial}{\partial w} \in \mathfrak{g}_0$, then by Lemma 9

$$\alpha(z) = \alpha_0 - \bar{\alpha}_1 z - \bar{\alpha}_0 A_{\chi^2}(z, 0), \qquad \beta(z) = \bar{\beta}_0 + 2i\bar{\alpha}_0 z.$$

We can thus embed $\mathfrak{g}_0 \hookrightarrow \mathbb{C} \times i\mathbb{R} \times \mathbb{R}$ via $X \mapsto (\alpha_0, \alpha_1, \beta_0)$. We also observe that the hypersurface A tangent to X is uniquely determined as the solution of

$$\big(\beta(z) - \bar{\beta}(\chi)\big)\big(1 + A^2\big) = i\big(\alpha(z)A_z(z, \chi) + \bar{\alpha}(\chi)A_\chi(z, \chi)\big).$$

We can read this equation as an ODE in either z or χ, which A solves with initial condition $A(0, \chi) = 0$ or $A(z, 0) = 0$. (We note that not every such A actually satisfies the reality relations.) An equivalent ODE for $B = \frac{1+iA}{1-iA}$ is given by

$$\big(\beta(z) - \bar{\beta}(\chi)\big)B = \alpha(z)B_z(z, \chi) + \bar{\alpha}(\chi)B_\chi(z, \chi).$$

Clearly $1 \leq \dim \mathfrak{g}_0 \leq 4$. For $\dim \mathfrak{g}_0 \geq 2$, observe that for any choice of $\alpha_0 \neq 0$ and α_1, and any choice of a real series $\sum_{j\geq 2} a_j t^j$, Lemma 12 shows that there exists a unique real function $A(z, \chi)$ with $A_{j,j} = a_{2j}$ and $\operatorname{Im} A_{j,j-1} = a_{2j-1}$ such that $X = (\alpha_0 + \alpha_1 z - \bar{\alpha}_0 A_{\chi^2}(z, 0))\frac{\partial}{\partial z} + 2i\bar{\alpha}_0 zw\frac{\partial}{\partial w}$ is tangent to $t = sA(z, \bar{z})$. On the other hand, for $\alpha_0 = 0$, $iz\frac{\partial}{\partial z} \in \mathfrak{g}_0$ just means that M is invariant under rotations in the z-variable. Our next result shows that $\dim \mathfrak{g}_0 \neq 3$:

Proposition 21 *If* $\dim \mathfrak{g}_0 \geq 3$, *then* $\dim \mathfrak{g}_0 = 4$, *and there is a unique* 1*-parameter family of hypersurfaces with* $\dim \mathfrak{g}_0 = 4$ *whose defining equations in normal form are given by*

$$w = \bar{w}\big(1 - C|z|^2\big)^{-\frac{2i}{C}}, \quad C \neq 0, \quad \textit{and for} \quad C = 0, \quad w = \bar{w}e^{2i|z|^2}.$$

Proof If $\dim \mathfrak{g}_0 \geq 3$, either $iz\frac{\partial}{\partial z} \in \mathfrak{g}_0$ or not; let us start by ruling out the latter possibility. After rotating, we can assume that we have $X, Y \in \mathfrak{g}_0$ which are given by

$$X = \big(1 - \varphi(z)\big)\frac{\partial}{\partial z} + 2izw\frac{\partial}{\partial w}, \qquad Y = i(1 + tz + \varphi)\frac{\partial}{\partial z} + 2zw\frac{\partial}{\partial w},$$

where we write $\varphi(z) = A_{\chi^2}(z, 0)$. Under the assumption that $\dim \mathfrak{g}_0 = 3$, we compute that

$$[X, Y] = i\big(t + (2 + tz)\varphi'(z) - \varphi(z)\big)\frac{\partial}{\partial z} \mod w\frac{\partial}{\partial w}.$$

But if $\dim \mathfrak{g}_0 = 3$, this means that necessarily $[X, Y] = tY$ modulo $w\frac{\partial}{\partial w}$; comparing the $\frac{\partial}{\partial z}$-coefficient of $[X, Y]$ and tY thus gives the following ODE for φ:

$$(2 + tz)\varphi'(z) - 2t\varphi(z) = t^2 z.$$

The solution $\varphi(z)$ of this ODE is $\varphi(z) = \frac{t^2}{4}z^2$. We now look at the tangency equation for this particular choice of α, β and see that it translates to

$$2i(z - \chi)B = \left(1 - \frac{t^2}{4}z^2\right)B_z + \left(1 - \frac{t^2}{4}\chi^2\right)B_\chi.$$

We know that if such a B exists, it is the unique solution of this equation with $B(z, 0) = 1$; making the ansatz $B(z, \chi) = f(z\chi)$ leads to the following equation for f:

$$2i(z - \chi)f(z\chi) = \left(1 - \frac{t^2}{4}z^2\right)\chi f'(z\chi) + \left(1 - \frac{t^2}{4}\chi^2\right)z f'(z\chi).$$

Extracting out of this power series equation terms of the form $z^{j+1}\chi^j$ on the one hand, and of the form $z^j\chi^{j+1}$ on the other hand, leads to one equation for f, namely,

$$2if(x) = \left(1 - \frac{t^2}{4}x\right)f'(x).$$

Solving this equation with $f(0) = 1$ thus gives the unique defining function for M to which X is tangent as $w = \bar{w}B(z, \bar{z})$ where $B(z, \chi) = f(z\chi)$; which is a contradiction since the real part of $iz\frac{\partial}{\partial z}$ is clearly tangent to M.

Thus we can safely assume that $iz\frac{\partial}{\partial z} \in \mathfrak{g}_0$ if $\dim \mathfrak{g}_0 \geq 3$. But then we definitely have (after possibly a rotation) the infinitesimal automorphisms

$$X = \big(1 - \varphi(z)\big)\frac{\partial}{\partial z} + 2izw\frac{\partial}{\partial w}, \qquad R = iz\frac{\partial}{\partial z},$$

whose commutator is given by $[R, X] = i(1 - z\varphi'(z) + \varphi(z))\frac{\partial}{\partial z}$ modulo $w\frac{\partial}{\partial w}$, which is linearly independent of X and R, and thus we have $\dim \mathfrak{g}_0 = 4$.

We can then choose a basis of the form

$$X = \big(1 - \varphi(z)\big)\frac{\partial}{\partial z} + 2izw\frac{\partial}{\partial w}, \qquad Y = i(1 + \varphi)\frac{\partial}{\partial z} + 2zw\frac{\partial}{\partial w}, \qquad R = iz\frac{\partial}{\partial z}.$$

The commutator of X and Y is given by $[X, Y] = 2i\varphi'(z)\frac{\partial}{\partial z} \mod w\frac{\partial}{\partial w}$, and thus $\varphi(z) = Cz^2$ for some $C \in \mathbb{R}$, and the commutator relation is given by $[X, Y] = 2CR$ modulo real multiples of $w\frac{\partial}{\partial w}$.

We now integrate (the real part of) X in order to find our hypersurfaces. They are defined by equations $w = \bar{w}B(|z|^2)$, and the ODE for B is $2iB(x) = (1 - Cx)B'(x)$, whose solution with $B(0) = 1$ is given by $B = (1 - Cx)^{-\frac{2i}{C}}$ for $C \neq 0$ and $B = e^{2ix}$ for $C = 0$, as claimed. A similar computation shows that we could equivalently have used Y, which shows that $\dim \mathfrak{g}(B)_0 = 4$. □

Remark 4 Hence, we have a complete description of the ruled hypersurfaces with a full group of possible transitive automorphisms, i.e., hypersurfaces M which satisfy that (M, p) is biholomorphically equivalent to (M, q) for $p, q \in E$. It is also interesting to compute the automorphisms themselves, which are given by

$$(z, w) \mapsto \begin{cases} (\frac{z+a}{1+C\bar{a}z}, w(1 + C\bar{a}z)^{\frac{2i}{C}}), & C \neq 0 \\ (z + a, we^{2iz\bar{a}}). \end{cases}$$

Remark 5 The hypersurfaces described above are all closely related. Let us write S_C for the hypersurface given by the equation $w = \bar{w}(1 - C|z|^2)^{-\frac{2i}{C}}$. For $r > 0$ the modification $(z, w) \mapsto (rz, w^{r^2})$ maps S_C into S_{r^2C}. Therefore, there are 3 "primitive" models in this case, S_{-1}, S_0, and S_1. Each of these can be written as a suitable modification of the sphere $\operatorname{Im} \eta = |\zeta|^2$. In the case of S_0, this modification is given by

$$\eta = \ln w, \qquad \zeta = z.$$

In the case of S_1, it is

$$\eta = \frac{w^{\frac{i}{2}} - 1}{w^{\frac{i}{2}} + 1}, \qquad \zeta = \frac{z}{w^{\frac{i}{2}} + 1},$$

with a similar formula for S_{-1}.

9 Sphere Blowups

In this section, we discuss ruled hypersurfaces M which are blowups of spheres. By this, we mean hypersurfaces for which there is a nontrivial map $H: (M, 0) \to (\mathbb{H}, 0)$, where $\mathbb{H} \subset \mathbb{C}^2_{(\zeta,\eta)}$ is given by $\operatorname{Im} \eta = |\zeta|^2$. In our computations, we will use the standard weights for (ζ, η): ζ has weight 1, η has weight 2. Since M is assumed to be ruled, H is necessarily of the form

$$H(z, w) = \big(f(z)w^{k+iT}, g(z)w^{2k}\big).$$

After a modification replacing w^k by w, we can assume that $k = 1$, and so $H = (f(z)w^{1+iT}, g(z)w^2)$. On the other hand, a simple computation reveals that in the preimage of $\mathbb{H}$ under such a map, we find a ruled hypersurface: If there is such an M, its defining function of the form $w = \bar{w}B(z, \bar{z})$ is determined by the equation

$$g(z)B(z, \chi)^2 - 2if(z)\bar{f}(\chi)B(z, \chi)^{1+iT} - \bar{g}(\chi) = 0. \qquad (30)$$

This equation has a unique solution B with $B(0,0)=1$, which automatically satisfies $B(z,\chi)=\bar{B}(\chi,z)$, and thus defines a real hypersurface, provided that

$$2g(0)-2i(1+iT)|f(0)|^2 \neq 0. \tag{31}$$

9.1 Distinguished Normalization—Defining Equations

In this subsection, we define a family of (pairwise inequivalent) hypersurfaces $M_{R,T}$ for $R, T \in \mathbb{R}$, which arise as sphere blowups. If M is assumed to be in distinguished normal form, the normality conditions for B are

$$B(z,0)=B(0,\chi)=1, \qquad B_\chi(z,0)=2iz, \qquad B_z(0,\chi)=2i\chi.$$

From the first condition we obtain that

$$g(z)=2if(z)\bar{f}_0+\bar{g}_0,$$

where as usual we write $f(z)=\sum_j f_j z^j$ and $g(z)=\sum_j g_j z^j$. To determine f, we take the χ-derivative of (30), substitute $B_\chi(z,0)=2iz$, and $g=2if\bar{f}_0+\bar{g}_0$, to obtain

$$f(z)=\frac{f_0\bar{f}_1+2z\bar{g}_0}{\bar{f}_1-2i(1-iT)z\bar{f}_0}.$$

In order for the parameters f_0, f_1, g_0 to fit with the definition of f and g, we need to have

$$g_0=\bar{g}_0+2if_0\bar{f}_0, \qquad f_1\bar{f}_1=2(\bar{g}_0+if_0\bar{f}_0)+2Tf_0\bar{f}_0,$$

i.e., $\operatorname{Im} g_0=i|f_0|^2$, and $2\operatorname{Re} g_0=|f_1|^2-2T|f_0|^2$. In particular, (31) holds for our (f,g) if and only if $f_1\neq 0$, which we shall assume from now on. However, the two complex parameters f_0, f_1 not all give rise to inequivalent hypersurfaces; and we still have some amount of freedom in the choice of blowup left, namely replacing f by λf for $|\lambda|=1$, or replacing z by λz for $|\lambda|=1$, and finally also replacing f by tf and g by $t^2 g$ for $0\neq t\in\mathbb{R}$. If we use these, we can require that

$$f_0=R\in\mathbb{R}_+, \qquad f_1=\lambda\in\mathbb{S}^1, \qquad g_0=\frac{1}{2}+R^2(i-T). \tag{32}$$

We then obtain

$$\begin{aligned} f(z)&=\frac{R+\lambda z(1-2R^2(i+T))}{1-2\lambda Rz(i+T)}, \\ g(z)&=\frac{1+2R^2(i-T)+2\lambda Rz(i-T)(1-2R^2(i+T))}{2(1-2\lambda Rz(i+T))}. \end{aligned} \tag{33}$$

We can now compute $B_{\chi^2}(z,0)$ from (30) with this data, setting $\lambda=1$, and see that

$$B_{\chi^2}(z,0)=-4\big(1+2i\big(T+3R^2\big(1+T^2\big)\big)\big)z^2+8iR\big(1+T^2\big)\big(1-2R(i+T)\big)z^3. \tag{34}$$

In particular, one can check that the blowups with parameters (R, T) and (R', T'), which we are going to denote by $M_{R,T}$ (and $M_{(R',T')}$, respectively), are not biholomorphically equivalent if $(R, T) \neq (R', T')$, because the associated $B_{\chi^2}(z, 0)$ are different.

9.2 Distinguished Normalization—Infinitesimal Automorphisms

Our goal in this subsection is the computation of $\mathfrak{hol}(M_{R,T})$. This will be used later, as we will show that ruled hypersurfaces with more than two infinitesimal automorphisms, not all of zero degree, are biholomorphically equivalent to one of the $M_{R,T}$.

In order to compute the infinitesimal automorphisms for $M_{R,T}$, we are going to use the infinitesimal automorphisms of $\mathbb{H}$ and pull them via $H = (f(z)w^{1+iT}, g(z)w^2)$. One computes that

$$w^{2+iT}\big(2f'(z)g(z) - (1+iT)f(z)g'(z)\big)\frac{\partial}{\partial\zeta} = 2g(z)w\frac{\partial}{\partial z} - g'(z)w^2\frac{\partial}{\partial w},$$

$$w^{2+iT}\big(2f'(z)g(z) - (1+iT)f(z)g'(z)\big)\frac{\partial}{\partial\eta} = -(1+iT)f(z)w^{iT}\frac{\partial}{\partial z} + f'(z)w^{1+iT}\frac{\partial}{\partial w}.$$

With our particular choice of H from above, we see that $2f'(0)g(0) - (1 + iT)f(0)g'(0) = (1 + R^2T)^3$.

The vector fields of degree 0 tangent to $\mathbb{H}$ are

$$i\zeta\frac{\partial}{\partial\zeta}, \quad \zeta\frac{\partial}{\partial\zeta} + 2\eta\frac{\partial}{\partial\eta}.$$

They both lift to infinitesimal automorphisms of $M_{R,T}$, the first one to

$$X_0 = \frac{2if(z)g(z)}{2f'(z)g(z) - (1+iT)f(z)g'(z)}\frac{\partial}{\partial z} - \frac{if(z)g'(z)}{2f'(z)g(z) - (1+iT)f(z)g'(z)}w\frac{\partial}{\partial w} \tag{35}$$

and the second one to

$$\frac{-2iTf(z)g(z)}{2f'(z)g(z) - (1+iT)f(z)g'(z)}\frac{\partial}{\partial z} + \frac{2f'(z)g(z) - f(z)g'(z)}{2f'(z)g(z) - (1+iT)f(z)g'(z)}w\frac{\partial}{\partial w} = w\frac{\partial}{\partial w} - TX_0.$$

Also the space of infinitesimal automorphisms of homogeneity 2 of $\mathbb{H}$ is 2-dimensional; they are explicitly given by

$$\big(2i\bar{a}\zeta^2 + a\eta\big)\frac{\partial}{\partial\zeta} + 2i\bar{a}\zeta\eta\frac{\partial}{\partial\eta}, \quad a \in \mathbb{C}$$

giving rise to

$$Y_a = \frac{2i\bar{a}(1-iT)f(z)^2g(z)w^{iT} + 2ag(z)^2w^{-iT}}{2f'(z)g(z) - (1+iT)f(z)g'(z)}w\frac{\partial}{\partial z}$$

$$+\frac{2i\bar{a}f(z)(f'(z)g(z)-f(z)g'(z))w^{iT}-ag(z)g'(z)w^{-iT}}{2f'(z)g(z)-(1+iT)f(z)g'(z)}w^2\frac{\partial}{\partial w}. \quad (36)$$

These are infinitesimal automorphisms only if $T=0$ (the coefficients branch at $w=0$ if $T\neq 0$).

Last, we have the homogeneous infinitesimal automorphism of degree 2 of $\mathbb{H}$ given by

$$\zeta\eta\frac{\partial}{\partial\zeta}+\eta^2\frac{\partial}{\partial\eta},$$

which translates to the vector field

$$X_2=\frac{(1-iT)f(z)g(z)^2}{2f'(z)g(z)-(1+iT)f(z)g'(z)}w^2\frac{\partial}{\partial z} + \frac{g(z)(f'(z)g(z)-f(z)g'(z))}{2f'(z)g(z)-(1+iT)f(z)g'(z)}w^3\frac{\partial}{\partial w}. \quad (37)$$

Of course, the homogeneous infinitesimal automorphisms of $\mathbb{H}$ of negative degree also lift along (zw^{1+iT}, w^2); but differently from the ones of positive degree; we obtain *singular* infinitesimal automorphisms in this setting. The infinitesimal automorphisms of degree -1 are given by

$$a\frac{\partial}{\partial\zeta}+2i\bar{a}\zeta\frac{\partial}{\partial\eta}, \quad a\in\mathbb{C},$$

giving rise to

$$Y_{a,-}=2\frac{ag(z)w^{-iT}+i\bar{a}(1+iT)f(z)^2w^{iT}}{2f'(z)g(z)-(1+iT)f(z)g'(z)}\frac{1}{w}\frac{\partial}{\partial z} + \frac{2i\bar{a}f(z)f'(z)w^{iT}-ag'(z)w^{-iT}}{2f'(z)g(z)-(1+iT)f(z)g'(z)}\frac{\partial}{\partial w}. \quad (38)$$

Last, the vector field $\frac{\partial}{\partial\eta}$, i.e., the homogeneous infinitesimal automorphism of $\mathbb{H}$ of degree -2 gives rise to

$$X_{2,-}=\frac{-(1+iT)f(z)}{2f'(z)g(z)-(1+iT)f(z)g'(z)}\frac{1}{w^2}\frac{\partial}{\partial z} + \frac{f'(z)}{2f'(z)g(z)-(1+iT)f(z)g'(z)}\frac{1}{w}\frac{\partial}{\partial w}. \quad (39)$$

Let us summarize the outcome of these computations. If $T=0$, we have that

$$\mathfrak{g}=\langle T, X_0, Y_1, Y_i, X_2\rangle_{\mathbb{R}}.$$

If on the other hand $T\neq 0$,

$$\mathfrak{g}=\langle T, X_0, X_2\rangle_{\mathbb{R}}.$$

This follows because $\dim \mathfrak{hol}(\mathbb{H}, 0) = 8$, and we have accounted for all of these (the three linearly independent singular vector fields giving rise to the part of negative homogeneity).

9.3 Tubular Normalization—Models

We can also construct tubular normal forms for the sphere blowups. These have the advantage of giving rise to different formulas (for the defining functions and infinitesimal automorphisms) which are better suited for some purposes.

In order to construct tubular normal forms for a sphere blown up via $H = (f(z)w^{1+iT}, g(z)w^2)$ we pull the infinitesimal automorphism $\zeta\eta\frac{\partial}{\partial\zeta} + \eta^2\frac{\partial}{\partial\eta}$ using H and require the outcome to be of the form $w^2\frac{\partial}{\partial z}$, i.e., we start with the equation

$$\frac{(1-iT)f(z)g(z)^2}{2f'(z)g(z)-(1+iT)f(z)g'(z)}w\frac{\partial}{\partial z} + \frac{g(z)(f'(z)g(z)-f(z)g'(z))}{2f'(z)g(z)-(1+iT)f(z)g'(z)}w^2\frac{\partial}{\partial w}$$
$$= w^2\frac{\partial}{\partial z}. \tag{40}$$

This leads to the following system of ODEs for $f(z)$, $g(z)$:

$$\frac{g(z)(f'(z)g(z)-f(z)g'(z))}{2f'(z)g(z)-(1+iT)f(z)g'(z)} = 0$$
$$\frac{(1-iT)f(z)g(z)^2}{2f'(z)g(z)-(1+iT)f(z)g'(z)} = 1.$$

The first equation gives $f = Kg$ for some $K \in \mathbb{C}\setminus\{0\}$, which simplifies the second equation to $g'(z) = g^2(z)$, and therefore

$$g(z) = \frac{1}{C-z}, \qquad f(z) = \frac{K}{C-z}, \quad C, K \in \mathbb{C}.$$

Replacing f by λf for $|\lambda| = 1$ we can require the $K \in \mathbb{R}_+$. Changing (f, g) to $(r^2 f, rg)$ for $r = 1/K$ we can assume that $f(z) = g(z) = \tilde{K}(C-z)^{-1}$. If we replace z by sz, we can therefore also assume that $\tilde{K} = 1$. With this choice, the denominator appearing in the expressions above is

$$2f'(z)g(z) - (1+iT)f(z)g'(z) = \frac{(1-iT)}{C^3} \neq 0.$$

Now the equation

$$g(z)\lambda^2 - 2if(z)\bar{f}(\chi)\lambda^{1+iT} - \bar{g}(\chi) = 0$$

has $\lambda = 1$ as a solution provided that $\operatorname{Im} C = -1$. We therefore replace C by $C - i$ with $C \in \mathbb{R}$ and note that

$$g(z)B(z,\chi)^2 - 2if(z)\bar{f}(\chi)B(z,\chi)^{1+iT} - \bar{g}(\chi) = 0$$

can be solved for $B(z,\chi)$ with $B(0,0)=1$ if $C+T\neq 0$. We are therefore left with the family of hypersurfaces $M_{C,T}$, where $C,T\in\mathbb{R}$, defined by the equation

$$(C+i-\chi)B(z,\chi)^2-2iB(z,\chi)^{1+iT}-(C-i-z)=0. \tag{41}$$

A simple computation shows that

$$\begin{aligned}B(z,0)=1&-\frac{1}{2(C+T)}z-\frac{C+T+i(1+T^2)}{8(C+T)^3}z^2\\&-\frac{3C^2+C(T^3+6iT^2+7T+6i)-2T^4+6iT^3-2T^2+6iT-3}{48(C+T)^5}z^3\\&+\cdots.\end{aligned}$$

We can now check that it is possible to express C and T in terms of the coefficients b_1,b_2,b_3 in $B(z,0)=\sum_j b_j z^j$. This means that $B(z,0)$ uniquely determines C and T; thus $M_{C,T}$ is biholomorphically equivalent to $M_{C',T'}$ if and only if $C=C'$ and $T=T'$.

9.4 Tubular Normalization—Infinitesimal Automorphisms

It is interesting to compute the infinitesimal automorphisms in the tubular normal coordinates; they are given by

$$\begin{aligned}X_0&=\left(2i\frac{C-i}{1-iT}-\frac{2i}{1-iT}z\right)\frac{\partial}{\partial z}-\frac{i}{1-iT}w\frac{\partial}{\partial w}\\Y_a&=\left(2i\bar{a}w^{iT}+\frac{2a}{1-iT}(C-i-z)w^{-iT}\right)w\frac{\partial}{\partial z}-\frac{a}{1-iT}w^{-iT}w^2\frac{\partial}{\partial w}\\X_2&=w^2\frac{\partial}{\partial z}.\end{aligned} \tag{42}$$

Similar formulas can be found for the singular automorphisms $X_{2,-}$ and $Y_{a,-}$.

9.5 Conclusion—Sphere Blowups

Let us summarize what we have computed in this section: There exists a real two-parameter family of real hypersurfaces arising as blowups of spheres (either $M_{R,T}$ or $M_{C,T}$ described above) with "many" symmetries. These families satisfy that for generic parameters, the automorphism group has dimension 3, while for the special parameter $T=0$, the dimension of the automorphism group is 5. We will later show that *any* ruled hypersurface with that many symmetries is already necessarily biholomorphically equivalent to one of the $M_{R,T}$ (or, equivalently, one of the $M_{C,T}$).

10 Hypersurfaces with Nonlinearizable Symmetries

In this section, we show that any hypersurface which supports more than 2 infinitesimal vector fields, not all of zero degree, is equivalent to one of the sphere blowups introduced in Sect. 9. We restrict ourselves to hypersurfaces which satisfy in $A=(1,1)$, since Sect.7 shows that this is the only setting which needs to be considered. The proof of Theorem 2 can be found at the end of the section.

Our first lemma settles the easy case, where we have an infinitesimal automorphism in $\mathfrak{g}^0_{>0}$. The proof follows from combining Lemmas 6 and 11, and the computation of the dimensions of the infinitesimal automorphisms of the blowups in Sect. 9.

Lemma 22 *Let M be a ruled hypersurface given in normal form by $t = sA(z,\bar{z})$, and assume that* in $A = (1,1)$. *If M has an infinitesimal automorphism $\alpha(z)w^\ell \frac{\partial}{\partial z} + \beta(z)w^{\ell+1}\frac{\partial}{\partial w}$ with $\ell > 0$ and $\alpha(0) = 0$, then M is given by a blowup of* $\operatorname{Im}\eta = |\zeta|^2$ *by a map of the form $\zeta = \sqrt{2\ell} z w^{\ell(1+2iT)}$, $\eta = \pm w^{2\ell}$. In that case* $\dim\mathfrak{g} = 3$ *if $T \neq 0$ and* $\dim\mathfrak{g} = 5$ *if $T = 0$.*

In order to handle the case where $\alpha(0) \neq 0$, we first need a preparatory lemma which gives an a priori restriction that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_k \oplus \mathfrak{g}_{2k}$, like Proposition 20 for the case of the degenerate tubes.

Lemma 23 *Let M be a ruled hypersurface given in normal form by $t = sA(z,\bar{z})$, with* in $A = (1,1)$, *and assume that*

$$X = \alpha(z)w^\ell \frac{\partial}{\partial z} + \beta(z)w^{\ell+1}\frac{\partial}{\partial w}, \qquad Y = \gamma(z)w^k \frac{\partial}{\partial z} + \delta(z)w^{k+1}\frac{\partial}{\partial w},$$

are infinitesimal CR automorphisms of A satisfying $\alpha(0) \neq 0$ and $\gamma(0) \neq 0$, $\ell \neq 0$, $k \neq 0$, which in addition satisfy $[X,Y] = 0$. Then $X = tY$ for some $t \in \mathbb{R}$ unless $\ell = 2k$. Furthermore, if $\ell = 2k$ we necessarily have $\beta \neq 0$.

Proof We note that since A is in normal form, Lemma 9 shows that α, β, γ, and δ satisfy the following equalities:

$$\begin{aligned}
\alpha(z) &= \alpha_0 + (\ell\beta_0 - \bar{\alpha}_1)z + \bar{\alpha}_0\big(2i\ell z^2 - A_{\chi^2}(z,0)\big)\\
\beta(z) &= \bar{\beta}_0 + 2i\bar{\alpha}_0 z\\
\gamma(z) &= \gamma_0 + (k\delta_0 - \bar{\gamma}_1)z + \bar{\gamma}_0\big(2ikz^2 - A_{\chi^2}(z,0)\big)\\
\delta(z) &= \bar{\delta}_0 + 2i\bar{\gamma}_0 z,
\end{aligned} \tag{43}$$

with $\alpha_0, \alpha_1, \gamma_0, \gamma_1 \in \mathbb{C}$, $\beta_0, \delta_0 \in \mathbb{R}$. To simplify the following computations, we note that we can assume that $\alpha_0 = 1$ after a change of coordinates of the form $z \mapsto \lambda z$ with $|\lambda| = 1$ and rescaling X.

Since $[X,Y] = 0$, we have that

$$\begin{aligned}
0 &= \alpha\gamma' - \gamma\alpha' + k\beta\gamma - \ell\delta\alpha\\
0 &= \alpha\delta' - \gamma\beta' + (k-\ell)\beta\delta.
\end{aligned} \tag{44}$$

We first observe that from the second equation in (44),

$$2i\bar{\gamma}_0 - 2i\gamma_0 + (k-\ell)\beta_0\delta_0 = 0, \tag{45}$$

and from the first equation

$$\gamma_1 - \gamma_0\alpha_1 + k\beta_0\gamma_0 - \ell\delta_0 = 0. \tag{46}$$

Thus, γ_1 is of the form prescribed in (43) if and only if

$$\gamma_0(\alpha_1 - k\beta_0) + \bar{\gamma}_0(\bar{\alpha}_1 - k\beta_0) = (k - 2\ell)\delta_0. \tag{47}$$

We combine this last equation with (45) and obtain the following set of equations for γ_0:

$$\begin{cases} (\alpha_1 - k\beta_0)\gamma_0 + (\bar{\alpha}_1 - k\beta_0)\bar{\gamma}_0 = (k - 2\ell)\delta_0 \\ \qquad\qquad 2i\gamma_0 - 2i\bar{\gamma}_0 = (k - \ell)\beta_0\delta_0. \end{cases} \tag{48}$$

Thus, γ_0 is uniquely determined (by δ_0) unless $\beta_0 = 0$ or $\ell = 2k$.

Let us first deal with the case $\beta_0 = 0$. We first show that under these assumptions, we necessarily also have $\delta_0 = 0$. To see this, normalize first by setting $\gamma_0 = 1$, and use (44) to see that α_0 is real and that α_1 is given by

$$\alpha_1 = \alpha_0(\gamma_1 - \ell\delta_0).$$

Thus, $\alpha_1 + \bar{\alpha}_1 = \ell\beta_0 = 0$ together with the fact that α_0 is real means that

$$0 = -2\ell\delta_0\alpha_0.$$

Since $\alpha_0 \neq 0$, we conclude that necessarily $\delta_0 = 0$. We now switch again to the normalization where $\alpha_0 = 1$. Now, we use the derivative of the first equation in (44),

$$0 = \alpha\gamma'' - \gamma\alpha'' + k\big(\beta'\gamma + \beta\gamma'\big) - \ell\big(\delta'\alpha + \delta\alpha'\big),$$

to see that

$$2\gamma_2 - \gamma_0 2\alpha_2 + k(2i\gamma_0 + \beta_0\gamma_1) - \ell(2i\bar{\gamma}_0 + \delta_0\alpha_1) = 0.$$

Substituting from (43) and writing $A_{\chi^2}(z, 0) = Cz^2 + \cdots$ with $C \in \mathbb{R}$, we thus obtain

$$0 = \bar{\gamma}_0(4ik - 2i\ell + 2C) - \gamma_0(4i\ell - 2ik + 2C) + k\beta_0\gamma_1 - \ell\delta_0\alpha_1. \tag{49}$$

Now, if $\beta_0 = \delta_0 = 0$, from (45) we have that γ_0 is real, so (49) implies that $k = \ell$. This implies that $X = tY$, as a straightforward computation using the differential equations in (44) shows.

In the remaining case, we have $\beta_0 \neq 0$, and $\ell \neq 2k$. Here we use a direct calculation to show that the commutation relations (44) imply that $k = \ell$, and as before, this implies that $X = tY$. Indeed, now the set of equations (48) has a unique solution (up to rescaling), and if we plug this γ_0 into (49), we obtain that the real part of the expression on the right-hand side is $(k^2 - \ell^2)\beta_0$, hence $k = \ell$. □

As we have seen, Lemma 22 reduces our problem to the setting of Lemma 23. This means that we only need to consider the setting where all CR automorphisms

$$\alpha(z)w^j \frac{\partial}{\partial z} + \beta(z)w^{j+1}\frac{\partial}{\partial w} \in \mathfrak{g}_j$$

satisfy $\alpha(0) \neq 0$; let us say that these automorphisms are of "type 1". Using this notion, Lemma 23 implies that if all the CR automorphisms of positive homogeneity are of type 1 then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_k \oplus \mathfrak{g}_{2k}$, with $\dim \mathfrak{g}_{2k} = 1$. Therefore, if $\dim \mathfrak{g} \geq 3$, we either have $\dim \mathfrak{g}_k \geq 1$ or $\dim \mathfrak{g}_0 \geq 2$. We will simplify our situation first by a modification of the form (z, w^k) as in Lemma 5, so that from now on we assume that $k = 1$.

Lemma 24 *Assume that M is in normal form, all CR automorphism of positive homogeneity are of type* 1, $\dim \mathfrak{g}_2 = 1$, *and that* $\dim \mathfrak{g}_1 \geq 1$. *Then there exists an $R > 0$ such that $M = M_{R,0}$.*

Proof We first normalize so that $X_2 \in \mathfrak{g}_2$ is given by

$$X_2 = \big(1 + (is + \beta_0)z + \big(4iz^2 - \varphi(z)\big)\big)w^2 \frac{\partial}{\partial z} + (\beta_0 + 2iz)w^3 \frac{\partial}{\partial w},$$

where $\varphi(z) = A_{\chi^2}(z, 0)$, and write $Y \in \mathfrak{g}_1$ as

$$Y = \big(\gamma_0 + (\delta_0 - \bar{\gamma}_1)z + \bar{\gamma}_0\big(2iz^2 - \varphi(z)\big)\big)w \frac{\partial}{\partial z} + (\delta_0 + 2i\bar{\gamma}_0 z)w^2 \frac{\partial}{\partial w}.$$

By assumption, we know that $\gamma_0 \neq 0$, and we also know from Lemma 23 that $\beta_0 \neq 0$. We thus obtain the following equations for γ_0 from the fact that $[X_2, Y] = 0$:

$$\begin{aligned} 2i\gamma_0 - 2i\bar{\gamma}_0 &= -\beta_0\delta_0 \\ is\gamma_0 - is\bar{\gamma}_0 &= -3\delta_0. \end{aligned} \tag{50}$$

We will thus distinguish the cases $\delta_0 = 0$ and $\delta_0 \neq 0$; let us start with the second one. In that case, compatibility of the equations (50) implies that $s = 6/\beta_0$, and γ_0 is of the form

$$\gamma_0 = x + i\frac{\beta_0\delta_0}{4}, \quad x \in \mathbb{R}.$$

Since from $[X_2, Y] = 0$ we have that $\gamma_1 = \gamma_0\alpha_1 - \beta_0\gamma_0 + \delta_0$, we can compute that

$$\gamma_1 = \frac{8\delta_0 - s\beta_0\delta_0}{4} + isx = \frac{\delta_0}{2} + isx.$$

With these formulas for γ_0 and γ_1, we now read the $\frac{\partial}{\partial z}$-component of the equation $[X_2, Y] = 0$ as an ODE for $\varphi(z)$. This ODE is given by

$$24z + 12\beta_0 z^2 - 6i\beta_0\varphi(z) + (2iz + \beta_0)\varphi'(z) = 0.$$

We solve for $\varphi(z)$ under the assumption that $\varphi(0) = 0$ and obtain that

$$\varphi(z) = \frac{4z^2(3\beta_0 + (2i + \beta_0^2)z)}{\beta_0^3}. \tag{51}$$

Hence X_2, and therefore M (up to a rotation in the z-variable) are uniquely determined by β_0 (which we can assume to be positive) and the fact that there exists a Y

as above with $[X_2, Y] = 0$. However, we already know a family of hypersurfaces with this property, namely our $M_{R,0}$ from Sect. 9.1; if we compute the X_2 for these, we see that we get $R\beta_0 = 1$. Hence, if we choose $R = \beta_0^{-1}$, we have that after a rotation $M = M_{R,0}$.

We are thus left with the case $\delta_0 = 0$. In that case, we have that $\gamma_0 \in \mathbb{R}$, and we will rescale Y so that $\gamma_0 = 1$. If we again write $\alpha_1 = is + \beta_0$, the formula $\gamma_1 = \alpha_1 - \beta_0$ obtained from $[X_2, Y] = 0$ implies that $\gamma_1 = is$. The equation for φ now becomes

$$i(\beta_0 s - 6) + 4sz + 12z^2 - 6i\varphi(z) + (\beta_0 + 2iz)\varphi'(z) = 0.$$

Thus, a solution to this equation with $\varphi(0) = 0$ forces that $s = 6/\beta_0$, and the solution $\varphi(z)$ with $\varphi(0) = 0$ is again given by (51) from which we conclude that M after a rotation is $M_{R,0}$ as in the preceding case. □

Lemma 25 *Assume that M is in normal form, all CR automorphisms of positive homogeneity are of type* 1, $\dim \mathfrak{g}_2 = 1$, *and that* $\dim \mathfrak{g}_0 \geq 2$. *Then there exists an $R > 0$ and a $T \in \mathbb{R}$ such that $M = M_{R,T}$.*

Proof We start as in the proof of Lemma 24 and normalize so that $X_2 \in \mathfrak{g}_2$ is given by

$$X_2 = \big(1 + (is + \beta_0)z + \big(4iz^2 - \varphi(z)\big)\big)w^2\frac{\partial}{\partial z} + (\beta_0 + 2iz)w^3\frac{\partial}{\partial w},$$

where $\varphi(z) = A_{\chi^2}(z, 0)$. We now assume that $Y \in \mathfrak{g}_0$ is given by

$$Y = \big(\gamma_0 + itz - \bar{\gamma}_0\varphi(z)\big)\frac{\partial}{\partial z} + (\delta_0 + 2i\bar{\gamma}_0 z)w\frac{\partial}{\partial w},$$

and that δ_0 is chosen in such a way that $[X_2, Y] = 0$ (this is possible since $\dim \mathfrak{g}_2 = 1$). Evaluating $[X_2, Y](0) = 0$, this means that the following equations have to be satisfied:

$$\begin{aligned} 2i(\bar{\gamma}_0 - \gamma_0) &= 2\beta_0\delta_0 \\ it - (is + \beta_0)\gamma_0 &= 2\delta_0. \end{aligned} \tag{52}$$

From the first one we conclude that $\gamma_0 = x + \frac{\beta_0\delta_0}{2}i$, which substituted into the second equation gives the system of equations

$$-x\beta_0 - 2\delta_0 - \frac{1}{2}s\beta_0\delta_0 = 0$$

$$t - sx - \frac{1}{2}\beta_0^2\delta_0 = 0.$$

Let us first observe that $\beta_0 \neq 0$: Indeed, if $\beta_0 = 0$, this implies that also $\delta_0 = 0$; i.e., $\gamma_0 = x \in \mathbb{R}$, and after rescaling to $x = 1$, we have $s = t$. One checks that this in turn implies that the coefficient of $zw^3\frac{\partial}{\partial z}$ in $[X_2, Y]$ is nonvanishing, i.e., $\beta_0 \neq 0$. We can now solve for x and t, obtaining

$$t = \delta_0 \frac{s^2\beta_0 - 4s - \beta_0^3}{2\beta_0}$$
$$x = \delta_0 \frac{4 - s\beta_0}{2\beta_0}.$$

One now checks that the equation $[X_2, Y] = 0$ translates to an ordinary differential equation for $\varphi(z)$, whose solution for $\varphi(0) = 0$ is given by

$$\varphi(z) = \frac{4z^2(s\beta_0 - 3)}{\beta_0^2} + \frac{4z^3(is\beta_0 + \beta_0^2 - 4i)}{\beta_0^3}.$$

Hence X_2, and thus M, is uniquely determined by the data s and β_0 and the fact that there exists a Y of class 1 and homogeneity 0. We already know a family of such hypersurfaces, namely the family $M_{R,T}$ from Sect. 9; we thus only need to check that after a rotation, we can obtain the parameters s and β_0 from a convenient choice of R and T. Now $\beta_0 + is$ is the coefficient of $zw^2\frac{\partial}{\partial z}$ in X_2, which is *invariant* under rotations.

We now assume that $\beta_0 > 0$ by multiplying X_2 by -1 if needed; this does not influence the preceding calculations.

If we now compute the coefficients of the homogeneous vector field

$$Z = (a_0 + a_1 z + \cdots)w^2\frac{\partial}{\partial z} + (b_0 + b_1 z)w^3\frac{\partial}{\partial w}$$

of degree 2 belonging to $\mathfrak{hol}(M_{R,T})$, we obtain

$$a_1 = i\frac{1 - iT}{4}\big(6R^2(1 + iT) - i\big)\big(1 - 4R^2T + 4R^4\big(1 + T^2\big)\big),$$
$$a_0 = \frac{1 - iT}{4}R\big(1 - 2R^2(T - i)\big)^2,$$

hence

$$\frac{a_1}{|a_0|} = i\frac{(1 - iT)}{R\sqrt{1 + T^2}}\big(6R^2(1 + iT) - i\big) = \frac{1}{\sqrt{1 + T^2}}\left(\frac{1}{R} + i\left(6R - \frac{T}{R} + 6RT^2\right)\right).$$

We thus need to show that the equations

$$\frac{1}{R\sqrt{1 + T^2}} = \beta_0$$
$$\frac{6R^2(1 + T^2) - T}{R\sqrt{1 + T^2}} = s$$

can be solved for R and T, given $s \in \mathbb{R}$ and $\beta_0 > 0$. Substituting $T = \tan\omega$ with $-\pi < 2\omega < \pi$, turns this into

$$\frac{\cos\omega}{R} = \beta_0$$
$$\frac{6R}{\cos\omega} - \frac{\sin\omega}{R} = s,$$

which is solved by

$$R^2 = \frac{\beta_0^2}{\beta_0^4 + (6 - s\beta_0)^2}, \qquad \omega = \arctan\left(\frac{6 - s\beta_0}{\beta_0^2}\right).$$

□

Lemma 26 *Let M be a ruled hypersurface, which is in addition circular and has an infinitesimal CR automorphism of class* 1*. Then M is given by the blowup of the sphere* $\operatorname{Im}\eta = |\zeta|^2$ *by the map* $\zeta = \sqrt{2\ell} z w^\ell$, $\eta = w^{2\ell}$.

Proof We use Lemma 5 in order to see that it is enough to consider an infinitesimal CR automorphism of the form

$$X = \alpha(z) w \frac{\partial}{\partial z} + \beta(z) w^2 \frac{\partial}{\partial w},$$

where $\alpha(0) \neq 0$; we are going to show that in this case, M is given by the blowup as claimed.

Since X is an infinitesimal CR automorphism of M and M is circular, we have that also $\tilde{X} = [X, iz\frac{\partial}{\partial z}]$ is an infinitesimal CR automorphism of M. However, a little computation shows that since $\alpha(z) = \alpha_0 + (\beta_0 - \bar{\alpha}_1)z + \bar{\alpha}_0(2iz^2 - A_{\chi^2}(z, 0))$, where $t = sA(z, \chi)$ defines M, and $\beta(z) = \bar{\beta}_0 + 2i\bar{\alpha}_0 z$, it follows that

$$\tilde{X} = \Big(i\alpha_0 - i\bar{\alpha}_0\big(2iz^2 - A_{\chi^2}(z, 0)\big)\Big)\frac{\partial}{\partial z} + 2\bar{\alpha}_0 z \frac{\partial}{\partial w}.$$

We thus assume that our vector field X satisfies $\alpha_1 = \beta_0 = 0$. Note that $A_{\chi^2}(z, 0) = Cz^2$ with $C \in \mathbb{R}$. If we write out the tangency condition in terms of B, we get

$$\alpha B B_z + \bar{\alpha} B_\chi + \beta B^2 - \bar{\beta} B.$$

Setting $B(z, \chi) = \varphi(z\chi)$, we see that

$$\alpha_0 \chi \left(\varphi\varphi' + (C - 2i)z\chi\varphi' - 2i\varphi\right) + \bar{\alpha}_0 z \big((C + 2i)z\chi\varphi\varphi' + \varphi' - 2i\varphi^2\big) = 0.$$

Since the series in the first summand only contains terms of the form $z^j\chi^{j+1}$ and the series in the second summand only terms of the form $z^{j+1}\chi^j$, they both need to vanish, i.e.,

$$\varphi\varphi' + (C - 2i)z\chi\varphi' - 2i\varphi = 0$$
$$(C + 2i)z\chi\varphi\varphi' + \varphi' - 2i\varphi^2 = 0.$$

We multiply the first equation by φ, subtract it from the second equation, and cancel φ' from the resulting equation to see that

$$\varphi^2 - 4iz\chi\varphi - 1 = 0,$$

and hence, $B(z,\chi) = 2iz\chi + \sqrt{1-4z^2\chi^2}$.

On the other hand, if we compute the defining equation of the blowup of the sphere with respect to the map in the claim of the lemma, it is given by the same formula. □

Proof of the list in Theorem 2 If $\dim \operatorname{Aut}(M,0) \geq 2$, then there exists a nontrivial homogeneous infinitesimal vector field $X \in \mathfrak{hol}(M,0)_k$ for some k which is not a multiple of $w\frac{\partial}{\partial w}$. After a modification as in Lemma 5 we can assume that $k=0$ or $k=2$. If $k=0$, either $X(0)=0$, in which case M is circular in normal coordinates by Lemma 8, if in $B=(1,m)$ with $m>1$, and by Lemma 9 for $m=1$. This means that a root of M is of the form in the first line of the table. If $X(0)\neq 0$, then M is a tube and Proposition 14 shows that it can be brought into the form in the second line of the table.

If $k=2$, then either $(w^{-2}X)(0) = \alpha_0$ is zero or not. If $\alpha_0 \neq 0$, we can use Proposition 15 and Lemma 5 to show that we can choose coordinates so that $X = w^2\frac{\partial}{\partial z}$. Computing the ODE from the tangency equation leads to the third line in the table. If on the other hand $\alpha_0 = 0$, then we use Lemmas 6, 5, and 11 to arrive at the form in the fourth line of the table.

Now, if $\dim \operatorname{Aut}(M,0) \geq 3$, we necessarily have in $B=(1,1)$ by Theorems 8 and 9. If $\dim \mathfrak{hol}(M,0)_0 \geq 3$ we apply Proposition 21 to see that $(M,0)$ is equivalent to one of the hypersurfaces in the fifth line of the table. The remaining list is obtained by applying Lemmas 22, 23, 24, 25, and 26. □

11 Classification of 1-Nonminimal Hypersurfaces

We now discuss how the normal form for ruled hypersurfaces can be used to classify 1-nonminimal hypersurfaces. Consider the action of a biholomorphism $H(z,w) = (f(z,w), g(z,w))$, decomposed in homogeneous terms as

$$f(z,w) = \sum_j f_j(z) w^j, \qquad g(z,w) = \sum_j g_j(z) w^{j+1},$$

with $f_0(z) = z$ and $g_0(z) = 1$ on the weighted homogeneous terms of order $\ell+1$ of the defining function of a 1-nonminimal hypersurface of the form $t = \varphi(z,\bar z, s)$. Let us again write $\varphi_s(z,\bar z,0) = A(z,\bar z)$ and assume that A is in normal form.

The transformation formula is computed to be

$$\tilde\varphi_{\ell+1} - \varphi_{\ell+1} = g_\ell(1+iA)^{\ell+1} - \overline{g_\ell}(1-iA)^{\ell+1} - \left(A_z f_\ell (1+iA)^\ell + A_{\bar z}\overline{f_\ell}(1-iA)^\ell\right), \tag{53}$$

modulo terms involving only lower-order homogeneous terms. Therefore, we are led to consider the family of "Chern–Moser operators"

$$\mathcal{L}_\ell \colon \mathcal{H}_\ell \to \mathcal{F}_{\ell+1}, \quad X = f_\ell(z) w^\ell \frac{\partial}{\partial z} + g_{\ell+1}(z) w^{\ell+1}\frac{\partial}{\partial w} \mapsto \big((\operatorname{Re} X)(t - sA)\big)|_{t=sA}$$

where $\mathcal{H}_\ell$ consists of homogeneous holomorphic vector fields of degree ℓ. Choosing a complement $\mathcal{N}_\ell$ of $(\mathcal{L}_\ell)$, we see that we can inductively solve for an $H = \sum H_\ell$ such that the transformed hypersurface has a defining equation of the form $t = sA(z,\bar{z}) + \tilde{\varphi}(z,\bar{z},s)$ with $\tilde{\varphi} \in \bigoplus_\ell \mathcal{N}_\ell =: \mathcal{N}$. H_ℓ is uniquely determined unless $\mathcal{L}_\ell$ has a nontrivial kernel, which is the case if and only if the model hypersurface $t = sA(z,\bar{z})$ has a nontrivial infinitesimal automorphism of degree ℓ.

This completes the proof of the formal classification given in Theorem 3. The claim about the biholomorphic classification in the real-analytic case follows from unpublished work of Juhlin (who proved convergence of formal maps between 1-nonminimal hypersurfaces) but can also be deduced from [6].

In order to prove Theorem 4, we need the following lemma.

Lemma 27 *Let $X = \alpha(z,w)\frac{\partial}{\partial z} + \beta(z,w)\frac{\partial}{\partial w} + w\frac{\partial}{\partial w}$ be a holomorphic vector field with $\alpha(z,0) = \beta(z,0) = \beta_w(z,0) = 0$, and denote by m the degree of the lowest-order homogeneous term of $X - w\frac{\partial}{\partial w}$. Then there exists a change of coordinates*

$$z = x + y^m F(x), \qquad w = y + y^{m+1} G(x)$$

such that in the new coordinates (x,y), the lowest-order homogeneous term of $X - y\frac{\partial}{\partial y}$ has degree strictly exceeding m.

Proof Let us start with the transformation formula for the differentials, which is given by

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{D}\left(\left(1 + (m+1)y^m G\right)\frac{\partial}{\partial x} - y^{m+1}G'\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial w} &= \frac{1}{D}\left(-my^{m-1}F\frac{\partial}{\partial x} + \left(1 + y^m F'\right)\frac{\partial}{\partial y}\right), \end{aligned}$$

where $D = (1 + y^m F')(1 + (m+1)y^m G) - my^{2m}FG' = 1 + y^m(F' + (m+1)G) - y^{2m}((m+1)F'G - mFG')$. Let us write $X = \alpha(z)w^m\frac{\partial}{\partial z} + \beta(z)w^{m+1}\frac{\partial}{\partial w} + w\frac{\partial}{\partial w}$ modulo terms of degree exceeding m.

Modulo such terms, we can therefore compute the transform of X as

$$X = y^m\left(\alpha(x) - mF(x)\right)\frac{\partial}{\partial x} + y^{m+1}\left(\beta(x) - (m+1)G(x)\right)\frac{\partial}{\partial y} + y\frac{\partial}{\partial y}$$

and see that with the appropriate choice of F and G we achieve that $X - y\frac{\partial}{\partial y}$ has no homogeneous terms of degree m or lower. □

The consequence of Lemma 27 which we are going to use in order to deduce Theorem 4 is actually quite interesting by itself:

Proposition 28 *Let $(M,0)$ be a germ of a 1-nonminimal hypersurface. Then $(M,0)$ is locally equivalent to a ruled hypersurface if and only if there exists an infinitesimal automorphism X of $(M,0)$ whose homogeneous part of degree 0 is of the form*

$$X_0 = \beta(z)w\frac{\partial}{\partial w}$$

with $\beta(0) \neq 0$.

Proof We first note that by a change of coordinates of the form $z = x$, $w = yg(x)$, we can assume that X_0 is actually given by $w\frac{\partial}{\partial w}$. An inductive application of Lemma 27 shows that we can construct a biholomorphism H such that in the new coordinates, X becomes $w\frac{\partial}{\partial w}$. Thus, in such coordinates, M is a ruled hypersurface. □

We can now end the proof of Theorem 4. If $\dim \operatorname{Aut}(M, 0) = 5$, there necessarily exists an infinitesimal automorphism X whose homogeneous part of degree 0 is of the form in Proposition 28. Thus, M is locally equivalent to a ruled hypersurface, and Theorem 2 applies.

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