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Levi degenerate hypersurfaces and Chern-Moser theory

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Chern-Moser theory for Levi nondegenerate hypersurfaces

Let $M \subseteq \mathbb{C}^n$ and $p \in M$ be a Levi nondegenerate point

– The model at p is a hyperquadric M_P:

$$\mathsf{Im}\, w = \sum_{j=1}^{n-1} \pm |z_j|^2, \quad (z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}.$$

– Symmetries of the model are well known, and control symmeries of M itself.

– The link between $Aut(M, p)$ and $hol(M_P, 0)$ is provided by the Chern-Moser operator

Aim: Extend the theory to hypersurfaces with polynomial models.

History:

Poincaré (1907): The local equivalence problem

– Poincaré - Moser ..... extrinsic approach

– Cartan - Chern -Tanaka ..... intrinsic approach

If p is Levi nondegenerate:

– In $\mathbb{C}^2$ solved by Cartan (1932)

– A complete solution in $\mathbb{C}^n$ given by Chern and Moser (1974)

If p is Levi degenerate:

– If the Levi form changes rank, the second approach is not available

– the extrinsic approach is available, using generalized Chern-Moser operators.

Chern-Moser normal form

$M \subseteq \mathbb{C}^2$ ... real analytic hypersurface

$p \in M$ ... a Levi nondegenerate point on M

(z, w) ... local coordinates, where $z = x + iy, \quad w = u + iv$

(centered at p, and such that $\{v = 0\}$ is tangent to M at p).

Near p, M is described as a graph of a real valued function

$$v = \Phi(x, y, u).$$

We will consider the Taylor expansion of Φ in terms of $z, \bar{z}, u$:

$$\Phi(z, \bar{z}, u) = \sum_{i+j+m \geq 2} a_{ijm} z^i \bar{z}^j u^m.$$

Normalizing the 2-nd order terms reveals the model hyperquadric.

The model hypersurface:

$$S = \left\{(z,w) \in \mathbb{C}^2 \mid v = |z|^2\right\}.$$

S is an unbounded version of the unit sphere $\{|z|^2 + |w|^2 = 1\}$.
It has a five dimensional group of local automorphisms $\mathcal{H}$ (linear fractional maps). We will write a general hypersurface as

$$v = |z|^2 + F(z, \bar{z}, u)$$

and consider the partial Taylor expansion of F in $z, \bar{z}$

$$F(z, \bar{z}, u) = \sum_{i,j=0}^{\infty} F_{ij}(u) z^i \bar{z}^j .$$

We will subject the defining equation to a general biholomorphic transformation

$$z^* = z + f(z,w)\,, \qquad w^* = w + g(z,w)$$

Theorem (Chern-Moser): *There exists a biholomorphic change of coordinates such that the new defining equation satisfies*

$$
\begin{aligned}
F_{j,0} &= 0\,, \qquad & j &= 0, 1, \dots\,, \\
F_{j,1} &= 0\,, \qquad & j &= 1, 2, 3, \dots\,, \\
F_{2,2} &= 0\,, \\
F_{3,3} &= 0\,, \\
F_{3,2} &= 0\,.
\end{aligned}
$$

This transformation is determined uniquely, up to a natural action of the symmetry group $\mathcal{H}$.

As a consequence, we obtain 2-jet determination for local automorphisms.

Refinement (Beloshapka, Kruzhilin): Either M is the sphere, or local automorphisms are linear in normal coordinates.

As a consequence, there is no Levi non-degenerate manifold with “interesting” symmetries beside the sphere.

The transformation formula

Substituting the transformation into $|z^*|^2 + F^* = v^*$ we get:

$$\Big|z + f\big(z, u + i(|z|^2 + F)\big)\Big|^2 + F^*\big(z + f(z, u + i(|z|^2 + F)), \overline{z + f(\dots)},$$
$$\text{Re } g(z, u + i(|z|^2 + F))\big) = |z|^2 + F + \text{Im } g\big(z, u + i(|z|^2 + F)\big) ,$$

where the argument of F is $(z, \bar{z}, u)$.

It gives an equality of two power series in $z, \bar{z}, u$.

In principle, it gives relations between various coefficients of F^* and F, f, g.

We give weights to the variables, weight $(z) = 1$, weight $(w) = 2$.

Separating the leading linear term leads to the **Chern-Moser operator:**

$$L(f, g) = \text{Re}\left(2\bar{z}f(z, u + i|z|^2) + ig(z, u + i|z|^2)\right) .$$

Proving the theorem involves analyzing the kernel and cokernel of L.

Levi degenerate manifolds

— New challenges, closer to algebraic, rather than to differential geometry.

— Starting with the work of J. J. Kohn the study of Levi degenerate manifolds has lead to major advances both in analysis and geometry
(e.g. microlocal analysis and subelliptic multiplier ideal sheaves).

Degrees of degeneracy:

– Levi flat: foliated by complex manifolds (all equivalent).

– Infinite type: contain a complex curve (not CR minimal).

– Finite type: type is the maximal order of contact of a complex curve with the hypersurface (Kohn 1972)

Levi-nondegenerate = type 2.

Finite type hypersurfaces

– Many models:

$$v = P(z, \bar{z})$$

where

$$P(z, \bar{z}) = \sum_{j=l}^{k-l} a_j z^j \bar{z}^{k-j},$$

with $a_j \in \mathbb{C}$ and $a_j = \overline{a_{k-j}}$.

Three qualitatively different cases: generic and two exceptional

– Circular

$$v = |z|^k$$

– Tubular

$$v = (Re\ z)^k$$

Normal forms for generic models:

Definition. F is in normal form if

$$\begin{aligned} F_{j0} &= 0, \qquad j = 1, 2, \ldots, \\ F_{k-l+j,l} &= 0, \qquad j = 0, 1, \ldots, \\ F_{2k-2l,2l} &= 0, \\ (F_{k-1}, P_z) &= 0, \end{aligned}$$

where

$$(P_{k-1}, P_z) = \sum_{j=1}^{k-2} F_{j,k-1-j}(j+1)\bar{a}_{j+1}.$$

Theorem (K., 2005). There is a formal transformation which takes F into normal form. It is determined uniquely up to a natural action of the symmetry group of the model.

The proof is based on a generalization of the Chern-Moser operator.

Degenerate manifolds in $\mathbb{C}^{n+1}$, $n > 1$.

Example: Let $l \in \mathbb{N}$ be arbitrary, and

$$M_P = \{(z_1, z_2, w) \in \mathbb{C}^3 \mid Im\ w \ = \ z_1 \bar{z}_2^l + {z_2}^l \bar{z}_1,\ l > 1\}$$

Then the vector field

$$Y = ai{z_2}^l \frac{\partial}{\partial z_1},\ a \in \mathbb{R},$$

gives a symmetry of M_P. It is determined by l-jets.

To generalize the Chern-Moser operator, we need again a (weighted) homogeneous model, which is provided by the concept of multitype due to Catlin.

Holomorphic vector fields

$$X = \sum_{j=1}^{n} f_j(z_1, \ldots, z_n) \frac{\partial}{\partial z_j}$$

provide infinitesimal automorphisms.

If M is given locally by $\{r = 0\}$, then X is an infinitesimal automorphism if

$$Re\ X(r) = 0$$

– The equivalence problem for holomorphic vector fields (Poincaré 1875)

If there is a vector field such that

$$X(r) = 0,$$

M is holomorphically degenerate.

Holomorphically degenerate = "of lower dimension", the symmetry group is infnite dimensional.

Polynomial models (joint work with *F. Meylan and D. Zaitsev*)

Let $\mathbb{C}_\nu[z]$ denote the space of holomorphic homogeneous polynomials in $z = (z_1, \ldots, z_n)$ of degree ν.

Theorem: Let $P(z, \bar{z})$ be a homogeneous polynomial without pluriharmonic terms of degree $d \geq 2$, such that M_P is holomorphically nondegenerate.
Then the Lie algebra $\mathfrak{g}$ of all germs of infinitesimal automorphisms of M_P at 0 admits the weighted grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/d} \oplus \mathfrak{g}_0 \oplus \bigoplus_{\tau=1}^{d-2} \mathfrak{g}_{\tau/d} \oplus \mathfrak{g}_{1-1/d} \oplus \mathfrak{g}_1.$$

Moreover, we have an explicit description of the graded components. E.g.

$$\mathfrak{g}_1 = \{\sum_j f^j(z) w \partial_{z_j} + a w^2 \partial_w : f^j \in \mathbb{C}_1[z],\ a \in \mathbb{R},\ \sum_j f^j(z) P_{z_j} = aP\}$$

– In particular, $\mathfrak{g}_1$ has real dimension less or equal to one

– Explicit characterization of hypersurfaces for which $\dim \mathfrak{g}_1 = 1$

As a consequence, we obtain an exact jet determination result.

Let M be given near p by

$$\mathsf{Im}\, w = P(z, \bar{z}) + o(|z|^d, \mathsf{Re}\, w).$$

Let $(f_1, f_2, \ldots, f_n, g)$ be the components of an automorphism of M.

Corollary: The automorphisms of M at p are uniquely determined by

(1) the complex tangential derivatives $\dfrac{\partial^{|\alpha|} f_j}{\partial z^\alpha}$ up to order $d-1$,

(2) the first and second order normal derivatives $\dfrac{\partial f_j}{\partial w}, \dfrac{\partial g}{\partial w}, \dfrac{\partial^2 g}{\partial w^2}$.

The jet determination result is sharp, as shown by the previous example.

Finite multitype

Let $p \in M$ be a point of finite Catlin multitype $(m_1, \ldots, m_n)$.

We can find coordinates $(z_1, \ldots, z_n, , w)$ with weight of z_j equal to $\mu_j = \frac{1}{m_j}$, weight of w equal to 1, such that M is locally given by

$$\operatorname{Im} w = P(z, \bar{z}) + F(z, \bar{z}, \operatorname{Re} w),$$

where P is a weighted homogeneous polynomial of weighted degree 1 and F has Taylor expansion with terms of weighted degree > 1.

To give the simplest example with unequal weights, consider

$$v = |z|^2 + |z|^6.$$

Then $\Lambda_M = (\frac{1}{2}, \frac{1}{6})$.

Denote by M_P the corresponding model

$$v = P(z, \bar{z}).$$

There is an explicit algorithm to compute the multitype.

Let $M \subseteq \mathbb{C}^4$ be given by

$$v = |z_1|^2 + 2Re\ z_1\bar{z}_2^3 + |z_2|^6 + Re\ z_1\bar{z}_2^4 + Re\ z_2^2\bar{z}_3^7.$$

The algorithm gives in five steps the multitype weights,

$$\Lambda_M = (\frac{1}{2}, \frac{1}{7}, \frac{5}{49}).$$

Theorem: The Lie algebra of infinitesimal automorphisms $\mathfrak{g} = \mathsf{aut}(M_P, 0)$ of M_P admits the weighted grading given by

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \bigoplus_{j=1}^{n} \mathfrak{g}_{-\mu_j} \oplus \mathfrak{g}_0 \oplus \bigoplus_{\eta \in E} \mathfrak{g}_\eta \oplus \mathfrak{g}_1. \tag{1}$$

where

$$E = \left\{ \sum_{j=1}^{n} k_j \mu_j; \quad k_j \in \mathbb{N} \cup \{-1\} \right\}.$$

We have an analogous explicit description of the graded components.

As a consequence, we obtain the following theorem that gives a sharp characterization of the automorphisms of M.

Corollary: The automorphisms of M at p are uniquely determined by their jets of weighted order 2.

Note that in the Levi nondegenerate case, when $\mu_j = \frac{1}{2}$, for all j, the first condition includes only the first order derivatives $\frac{\partial f_j}{\partial z_i}$.

Hence we recover the sharp result from the work of Chern and Moser.

Theorem: If M is Levi nondegenerate at p, its automorphisms are uniquely determined by the following partial derivatives

(1) the first order complex tangential derivatives $\dfrac{\partial f_j}{\partial z_i}$ for $i, j = 1, \ldots, n$

(2) the first and second order normal derivatives $\dfrac{\partial f_j}{\partial w}$ for $j = 1, \ldots, n$, $\dfrac{\partial g}{\partial w}$, $\dfrac{\partial^2 g}{\partial w^2}$.

The generalized Chern-Moser operator

$$L(f,g) = Re\left\{ ig(z,u+iP(z,\bar{z})) + 2\sum_{j=1}^{n} \frac{\partial P}{\partial z_j} f^j(z,u+iP(z,\bar{z})) \right\}.$$

where $f = (f^1, \ldots, f^n)$.

<u>Basic fact:</u>

The pair (f,g) lies in the kernel of L if and only if the vector field

$$Y = \sum_{j=1}^{n} f^j(z,w)\frac{\partial}{\partial z_j} + g(z,w)\frac{\partial}{\partial w}$$

lies in $hol(M_P, 0)$, the Lie algebra of infinitesimal local automorphisms of M_P at p.

Indeed, we have

$$
\begin{aligned}
&Re\, Y(v-P)_{|M_P} = \\
&\frac{1}{2} Re\, \left\{ ig\left(z, u+iP(z,\bar z)\right) + 2\sum_{j=1}^{n} \frac{\partial P}{\partial z_j} f^j\left(z, u+iP(z,\bar z)\right) \right\} = \frac{1}{2} L(f,g).
\end{aligned}
$$

First, using weights and the Chern-Moser operator we can reduce the weighted jet determination problem from $Aut(M, p)$ to $hol(M_P, 0)$.

The weights introduce a natural grading on $hol(M_P, 0)$ in the following sense. We write $hol(M_P, 0)$ as

$$hol(M_P, 0) = \oplus_{\mu+1\in\Theta} G_\mu,$$

where G_μ consists of weighted homogeneous vector fields of weight μ.

Observe that each weighted homogeneous component $X_\mu \in G_\mu$ of $X \in hol(M_P, 0)$ lies also in $hol(M_P, 0)$, since $v - P$ is weighted homogeneous.

Rigid vector fields

Definition: Let X be a vector field in $hol(M_P, 0)$ of the form

$$Y = \sum_{j=1}^{n} f^j(z,w)\frac{\partial}{\partial z_j} + g(z,w)\frac{\partial}{\partial w}$$

We say that X is rigid if $f_1, \ldots, f_n, g$ are all independent of the variable w.

Every model has a priori two symmetries: the rigid vector field W of homogeneous weight -1, given by

$$W = \frac{\partial}{\partial w}$$

and the (nonrigid) weighted Euler field E of homogeneous weight 0

$$E = \sum_{j=1}^{n} \mu_j z_j \frac{\partial}{\partial z_j} + w \frac{\partial}{\partial w}.$$

By the above observation, we may restrict ourselves to weighted homogeneous vector fields. The starting point is a jet determination result for rigid vector fields.

Proposition: Let M_P be holomorphically nondegenerate, and let $X \in hol(M_P, 0)$ be a nonzero rigid vector field given by

$$\sum_{j=1}^{n} f_j(z_1, \dots z_n) \frac{\partial}{\partial z_j},$$

where f_j are germs of holomorphic functions at p. Then the coefficients f_j are sums of monomials of weighted degree less than one.

Lemma: Let $X \in hol(M_P, 0)$ be a weighted homogeneous vector field, and let $W \in hol(M_P, 0)$ be as above. There exists an integer $l \geq 1$, and a rigid vector $Y \in hol(M_P, 0)$ such that $[\dots[[X; W]; W]; \dots]; W] = Y$, where the string of brackets is of length l.

Note that the effect of taking the bracket of X with W is simply differentiation of the coefficient with respect to w.

We say that $X \in hol(M_P, 0)$ is an l-integration of a rigid vector $Y \in hol(M_P, 0)$ if the string of brackets described in the above lemma is of length l.

Remark: By the above lemma, the general case will be reduced to the rigid case by taking sufficiently many commutators with the vector field W.

Hence the problem reduces to:

(i) describing rigid vector fields

(ii) analysing to what extent rigid fields can be "integrated".

By the previous result, we can divide homogeneous rigid vector fields into three types:

(i) *Shifts* are defined as the vector fields of weighted degree less than zero.

(ii) *Rotations* are defined as the vector fields of weighted degree zero.

(iii) *Generalized rotations* are defined as vector fields of weighted degree bigger than zero and less than one.

Nonintegrability of rotations

Proposition: Let $X \in hol(M_P, 0)$ be a vector field of the form

$$X = \sum_{j=1}^{n} f^j(z) \, \frac{\partial}{\partial z_j},$$

where the weight of f^j is μ_j. Then there exists no vector field Y coming from $hol(M_P, 0)$ such that

$$[Y, W] = X.$$

Proof: Integrating the coefficients of X with respect to w, we obtain

$$Y = w \sum_{j=1}^{n} f^j(z) \, \frac{\partial}{\partial z_j} + \sum_{j=1}^{n} \phi_j(z) \frac{\partial}{\partial z_j} + \phi(z) \frac{\partial}{\partial w}$$

Hence $Y \in hol(M_P, 0)$ is equivalent to

$$Re\, Y(P - v) = Re \left(2\sum_{j=1}^{n} f^j(z) \frac{\partial P}{\partial z_j}(u + iP(z,\bar{z})) + 2\sum_{j=1}^{n} \frac{\partial P}{\partial z_j}\phi_j(z) + i\phi(z) \right) =$$

$$= Re \sum_{j=1}^{n} f^j(z) \frac{\partial P}{\partial z_j} iP(z,\bar{z}) + 2\sum_{j=1}^{n} \frac{\partial P}{\partial z_j}\phi_j(z) - Im\, \phi(z) = 0,$$

where we have used $Re\, X(P - v) = 0$. The first two summands contains only mixed terms, while the third one is pluriharmonic, so $\phi(z) = 0$. Hence

$$-P(z,\bar{z}) Im\, X(P) + 2Re \sum_{j=1}^{n} \left(\frac{\partial P}{\partial z_j}\phi_j(z) \right) = 0.$$

By assumption, we may rewrite it as

$$-P(z,\bar{z}) X(P) + 2Re \sum_{j=1}^{n} \left(\frac{\partial P}{\partial z_j}\phi_j(z) \right) = 0.$$

The result now follows from

Lemma: Let $P = P(z, \bar{z})$ be a real homogeneous polynomial, X be a rigid vector field of weight 0 satisfying

$$(Re\ X)P = 0$$

and Z be a rigid holomorphic vector field satisfying the "main equation"

$$(Re\ Z)P + (Im\ X)P^2 = 0.$$

Then $[X, Z] = 0$ and $X(P) = 0$.

Nonintegrability of generalized rotations

The same result is obtained for generalized rotations. We have

Proposition: Let $P = P(z, \bar{z})$ be a real homogeneous polynomial, X be a rigid vector field of weight bigger than 0 satisfying

$$(Re\ X)P = 0$$

and Y be a vector field satisfying

$$(Re\ Y)P + (Im\ X)P^2 = 0.$$

Then $X(P) = 0$.

Integration of transversal shift

1-integration of W gives E, modulo rotations.

2-integration:

Proposition: Let $W \in hol(M_P, 0)$ be as above. Then there is a vector field lying in $hol(M_P, 0)$ that is a 2-integration of W if and only if M_P admits a complex reproducing field.

Nonexistence of other fields

– W cannot be integrated 3 times

– non-transversal shift cannot be integrated 2 times