INVESTMENTS IN EDUCATION DEVELOPMENT

Some Trends in Algebra

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Obstruction theory for the representability of cohomological functors

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Joint work with Fernando Muro

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Theorem [Adams 1971]

Let $\mathcal{S}^f \subset \mathcal{S}$ be the homotopy category of finite spectra.

$$\begin{aligned} H\colon \{\mathcal{S}^f\}^{\mathrm{op}} &\longrightarrow \mathrm{Ab} \\ \textstyle\prod &\longmapsto \textstyle\prod \\ \text{fiber seq.} &\longmapsto \text{long exact seq.} \end{aligned} \quad \Rightarrow \quad H \cong \mathcal{S}(-,X)|_{\mathcal{S}^f}$$

$$\mathcal{S}(-,X)|_{\mathcal{S}^f} \xrightarrow{\eta} \mathcal{S}(-,Y)|_{\mathcal{S}^f} \quad \Rightarrow \quad \eta = \mathcal{S}(-,g)|_{\mathcal{S}^f} \text{ for } X \xrightarrow{g} Y$$

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- Representability of cohomology theories [Brown 1962]:

$$H\colon \mathcal{S}^{\mathrm{op}} \longrightarrow \mathrm{Ab} \text{ cohomology theory} \Rightarrow H \cong \mathcal{S}(-,X)$$

- Representability of homology theories (via Spanier–Whitehead duality):

$$H\colon \mathcal{S} \longrightarrow \mathrm{Ab} \text{ homology theory} \Rightarrow H \cong \mathcal{S}(\mathbb{S}, X \wedge -)$$

- Brown representability for the dual [Neeman 1998]:

$$\begin{array}{rcl} H\colon \mathcal{S} & \longrightarrow & \mathrm{Ab} \\ \prod & \longmapsto & \prod \\ \text{fiber seq.} & \longmapsto & \text{long exact seq.} \end{array} \quad \Rightarrow \quad H \cong \mathcal{S}(X,-)$$

$\mathscr{T}$ satisfies α-Adams representability for objects ARO_α

$$H\colon \{\mathscr{T}^\alpha\}^{\mathsf{op}} \longrightarrow \mathsf{Ab} \quad \text{cohomological}$$
$$\textstyle\coprod_{<\alpha} \longmapsto \prod$$

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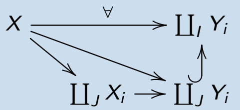
- $\mathscr{T}$ is a triangulated category with coproducts (additive, $\Sigma\colon \mathscr{T} \to \mathscr{T}$ equivalence, family of exact triangles $X \to Y \to Z \to \Sigma X$).
- $H\colon \mathscr{T}^{\mathrm{op}} \to \mathrm{Ab}$ is cohomological if it sends triangles to long exact sequences.

$$X \to Y \to Z \to \Sigma X \quad \mapsto \quad \cdots H(\Sigma^{-1}Z) \leftarrow H(X) \leftarrow H(Y) \leftarrow H(Z) \leftarrow H(\Sigma X) \cdots$$

- $\mathscr{T}^\alpha$ the full subcategory of α-compact objects for a regular cardinal α.



$\mathrm{card}(J) < \alpha$, X_i α-compact

Outline of the talk

1. Motivations
2. Results
3. Obstruction theory

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We define an obstruction theory for the representability of objects and morphisms in the subcategory $\mathrm{Mod}_\alpha(\mathscr{T}^\alpha) \subset \mathrm{Mod}(\mathscr{T}^\alpha)$ with objects

$$\{\mathscr{T}^\alpha\}^{\mathrm{op}} \longrightarrow \mathrm{Ab}$$
$$\coprod_{<\alpha} \longmapsto \prod$$

- Restricted Yoneda functor $\mathrm{S}_\alpha : \mathscr{T} \longrightarrow \mathrm{Mod}_\alpha(\mathscr{T}^\alpha)$, $X \longmapsto \mathscr{T}(-, X)|_{\mathscr{T}^\alpha}$
- $\mathrm{ARO}_\alpha \Leftrightarrow$ Cohomological functors in $\mathrm{Mod}_\alpha(\mathscr{T}^\alpha) \subset \mathrm{Ess.\ Im}(\mathrm{S}_\alpha)$
- $\mathrm{ARM}_\alpha \Leftrightarrow \mathrm{S}_\alpha$ is full

$\mathscr{T}$ a triangulated category with coproducts & α a regular cardinal.

- Generated by a set of objects S:

$$\mathscr{T}(s, X) = 0 \ \forall s \in S \Rightarrow X = 0$$

- α-compactly generated: Has coproducts & generated by a set of α-compact objects.
- Well generated: α-compactly generated for some α.

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Examples

1. Ho(Sp) is generated by finite spectra $\mathrm{Ho(Sp)}^{\aleph_0}$.
2. $\mathcal{M}$ stable combinatorial model category $\Rightarrow$ Ho($\mathcal{M}$) well generated [Rosický 2005].
3. D(R) is generated by complexes quasi-isomorphic to bounded complexes of finitely generated projective R-modules $\mathrm{D}(R)^{\aleph_0}$.
4. $\mathcal{A}$ Grothendieck category $\Rightarrow$ D($\mathcal{A}$) well generated [Alonso, Jeremías & Souto 2000].
5. $\mathcal{S} \subset \mathscr{T}$ localizing subcategory of a well generated $\Rightarrow \mathscr{T}/\mathcal{S}$ well generated [Thomason localization theorem, Neeman 2001].

Theorem [Neeman 1997]

$\mathscr{T}$ $\aleph_0$-compactly generated & $\mathrm{card}\,\mathscr{T}^{\aleph_0} \leq \aleph_0$ $\Rightarrow$ $\mathscr{T}$ satisfies $\mathrm{ARO}_{\aleph_0}$ & $\mathrm{ARM}_{\aleph_0}$

Theorem [Christensen, Keller & Neeman 2001]

$\mathrm{D}(\mathbb{C}\langle X, Y\rangle)$ satisfies $\mathrm{ARM}_{\aleph_0}$ $\Leftrightarrow$ Continuum Hypothesis

Theorem [Neeman 2001]

$\mathscr{T}$ is well generated, then

- $\mathscr{T} = \bigcup_\alpha \mathscr{T}^\alpha$ and
- $\mathscr{T}$ satisfies Brown representability.

$\mathscr{T}$ well generated $\overset{?}{\Rightarrow}$ $\mathscr{T}$ satisfies α-ARO & α-ARM for α large enough

Theorem [Neeman 2009]

$\mathscr{T}$ satisfies ARO_α & ARM_α $\Rightarrow$ $\mathscr{T}$ & $\mathscr{T}^{\mathrm{op}}$ satisfy Brown rep.

$\mathscr{T}$ well generated $\overset{?}{\Rightarrow}$ $\mathscr{T}^{\mathrm{op}}$ satisfies Brown representability

Theorem [M. & R.]

ARO_α	$\Leftarrow$	$\mathrm{pd}(H) \leq 2$ in $\mathrm{Mod}_\alpha(\mathscr{T}^\alpha)$ $\forall H$ cohomological
$\Uparrow$		$\Uparrow$
ARO_α & ARM_α	$\Leftrightarrow$	$\mathrm{pd}(H) \leq 1$ in $\mathrm{Mod}_\alpha(\mathscr{T}^\alpha)$ $\forall H$ cohomological
$\Downarrow$		$\Downarrow$
ARM_α	$\Leftrightarrow$	$\mathrm{pd}\big(\mathscr{T}(-,X)_{\mid \mathscr{T}^\alpha}\big) \leq 1$ in $\mathrm{Mod}_\alpha(\mathscr{T}^\alpha)$ $\forall X$ in $\mathscr{T}$

Theorem [M. & R.]

$\alpha \leq \aleph_n$, $\mathrm{card}\,\mathscr{T}^\alpha \leq \aleph_n$ $\Rightarrow$ $\sup\{\mathrm{pd}(F) \mid F \text{ cohomological in } \mathrm{Mod}_\alpha(\mathscr{T}^\alpha)\} \leq n+1$

Corollary

$\mathscr{T}$ is α-compactly generated card $\mathscr{T}^{\alpha} \leq \aleph_1$ with $\alpha = \aleph_0$ or $\aleph_1$	$\Rightarrow$	$\mathscr{T}$ satisfies ARO_{α}
$\mathscr{T}$ is $\aleph_0$-compactly gen. card $\mathscr{T}^{\aleph_0} \leq \aleph_0$	$\Rightarrow$	$\mathscr{T}$ sat. $\mathrm{ARO}_{\aleph_0}$ & $\mathrm{ARM}_{\aleph_0}$ [Christensen, Keller & Neeman]

Examples that satisfy $\mathrm{ARO}_{\aleph_1}$ assuming the continuum hypothesis

1. The stable homotopy category Ho(Sp).
2. The derived category of a ring $\mathrm{D}(R)$ with card $R \leq \aleph_1$.
3. The homotopy category of complexes of injective R-modules $\mathrm{K}(R\text{-Inj})$ with R noetherian and card $R \leq \aleph_1$.
4. The homotopy category of complexes of projective R-modules $\mathrm{K}(R\text{-Proj})$ with card $R \leq \aleph_1$.
5. The derived category of sheaves on a connected paracompact manifold $\mathrm{D}(Sh/M)$.
6. The stable motivic homotopy theory $\mathcal{SH}(S)$ over a noetherian scheme of finite Krull dimension $S = \bigcup_{i \in I} \mathrm{Spec}(R_i)$ with card $R_i \leq \aleph_1$ for all $i \in I$.

Theorem [M. & R.]

- Let R be right α-coherent & $\alpha > \aleph_0$.

  $\boxed{\mathrm{D}(R) \text{ satisfies } \mathrm{ARM}_\alpha} \Rightarrow \boxed{\mathrm{Pgldim}_\alpha(R) \leq 1}$

- Let R be an hereditary ring.

  $\boxed{\mathrm{D}(R) \text{ satisfies } \mathrm{ARO}_\alpha} \Leftrightarrow \boxed{\mathrm{Pgldim}_\alpha(R) \leq 2}$

  $\boxed{\mathrm{D}(R) \text{ satisfies } \mathrm{ARM}_\alpha} \Leftrightarrow \boxed{\mathrm{Pgldim}_\alpha(R) \leq 1}$

- α-coherent: R-modules with $< \alpha$ generators have $< \alpha$ relations.
- Hereditary: global projective dimension ≤ 1.
- α-pure global dimension: $\mathrm{Pgldim}_\alpha(R) \leq n$ if for any right R-module M, there is a sequence

$$0 \to P_n \to \cdots \to P_1 \to M \to 0$$

where each P_i is a retract of a direct sum of right R-modules with $< \alpha$ generators & $< \alpha$ relations and

$$0 \to \hom_R(Q, P_n) \to \cdots \to \hom_R(Q, P_1) \to \hom_R(Q, M) \to 0$$

is exact for all right R-modules Q with $< \alpha$ generators & $< \alpha$ relations.

R α-coherent & $\mathrm{Pgldim}_\alpha(R) > 1$ $\Rightarrow$ $\mathrm{D}(R)$ does not satisfy ARM_α

Computations of lower bounds to α-pure projective dimensions are given in [Baer & Lenzing 1982] for $\alpha = \aleph_0$, in [Braun & Göbel 2012] for $R = \mathbb{Z}$ & in [Bazzoni & Šťovíček 2013].

Corollary

ARM_α is not satisfied for rings R & cardinals α as indicated:

1. $R = \mathbb{Z}$ & $\alpha > \aleph_0$.
2. Let k be an uncountable field and α any regular cardinal or k a countable field & $\alpha > \aleph_0$.
    - $R = k[x, y]$.
    - R the path algebra of a finite quiver without oriented cycles which is not a Dynkin quiver.
3. $R = k[[x, y]]$ for any field k & any regular cardinal α.

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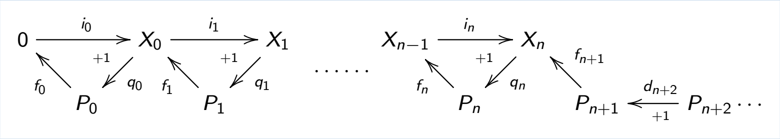
R hereditary, $\mathrm{Pgldim}_\alpha(R) > 2$ & $\alpha > \aleph_0$ $\Rightarrow$ $\mathrm{D}(R)$ does not satisfy ARO_α

We do not known any ring with this property

Definition (after [Benson, Krause & Schwede 2004] for $\alpha = \aleph_0$)

An *n-truncated Postnikov system* $(X_{\leq n}, P_*)$, $n \geq 0$, is a diagram in $\mathscr{T}$



- triangles are exact
- $f_{n+1} d_{n+2} = 0$ (cocycle condition)
- the restricted Yoneda functor $S_\alpha(X) = \mathscr{T}(-, X)_{|\mathscr{T}^\alpha}$ maps

$$P_0 \xleftarrow[+1]{q_0 f_1} P_1 \longleftarrow \cdots \longleftarrow P_n \xleftarrow[+1]{q_n f_{n+1}} P_{n+1} \xleftarrow[+1]{d_{n+2}} P_{n+2} \longleftarrow \cdots$$

to an exact sequence of projective objects.

- $\mathbf{Post_n}$ the category of *n*-truncated Postnikov systems.
- $\mathbf{Post_n^{\simeq}}$ the category of *n*-truncated Postnikov systems up to homotopy.

$(\psi_{\leq n}, \varphi_*) \simeq (\bar{\psi}_{\leq n}, \bar{\varphi}_*) \Leftrightarrow \psi_k - \bar{\psi}_k$ factors through $f_{k+1} \colon P_{k+1} \to X_k$, $0 \leq k \leq n$

- Postnikov resolution: $(\mathrm{Hocolim}_n X_n, X_*, P_*)$ with (X_*, P_*) in $\lim_n \mathrm{Post}_n^{\simeq}$.
- $\mathbf{Pres_\infty^{\simeq}}$ the category of Postnikov resolutions.

Theorem [M. & R.]

- There is a sequence of exact sequences of categories, $n \geq 0$,

$$\begin{array}{ccccccc} & & & & \mathrm{Ext}_{\alpha,\mathscr{T}}^{n+3,-1-n} & & \\ & & & & \uparrow \kappa_n & & \\ \mathrm{Ext}_{\alpha,\mathscr{T}}^{n+1,-1-n} & \xrightarrow{\imath_n} & \mathrm{Post}_{n+1}^{\simeq} & \xrightarrow{\text{trunc.}} & \mathrm{Post}_n^{\simeq} & \xrightarrow{\theta_n} & \mathrm{Ext}_{\alpha,\mathscr{T}}^{n+2,-1-n} \end{array}$$

- There exists an essentially unique functor

$$\Psi\colon \mathscr{T} \longrightarrow \mathrm{Pres}_\infty^{\simeq}$$

  - additive, full and essentially surjective,
  - $\ker \Psi$ is the ideal $\mathscr{I}^\infty$ of ∞-phantom maps, and
  - $\mathscr{I}^\infty$ is a square zero ideal.
  - The restricted Yoneda functor $S_\alpha(X) = \mathscr{T}(-,X)|_{\mathscr{T}^\alpha}$ factors as

$$\begin{array}{ccccc} \mathscr{T} & \xrightarrow{\Psi} & \mathrm{Pres}_\infty^{\simeq} & \longrightarrow & \mathrm{Post}_0^{\simeq} \simeq \mathrm{Mod}_\alpha(\mathscr{T}^\alpha) \\ X & \longmapsto & (\mathrm{Hocolim}_n X_n, X_*, P_*) & \longmapsto & H_* S_\alpha(P_*) \end{array}$$

- $\kappa_n(X_{\leq n}, P_*) \in \mathrm{Ext}_{\alpha,\mathscr{T}}^{n+3,-1-n}(H_0 S_\alpha(P_*), H_0 S_\alpha(P_*))$: Obstruction to extend an n-truncated Postnikov system $(X_{\leq n}, P_*)$ to an $(n+1)$-truncated one.
- $\theta_n(\psi_{\leq n}, \varphi_*) \in \mathrm{Ext}_{\alpha,\mathscr{T}}^{n+2,-1-n}(H_0 S_\alpha(P_*), H_0 S_\alpha(Q_*))$: Obstruction to extend an n-truncated morphism $(\psi_{\leq n}, \varphi_*)$ between $(n+1)$-truncated Postnikov systems to an $(n+1)$-truncated one.