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TRANSFINITE ADAMS REPRESENTABILITY

FERNANDO MURO AND ORIOL RAVENTÓS

Abstract. We consider the following problems in a well generated triangulated category $\mathscr{T}$. Let α be a regular cardinal and $\mathscr{T}^\alpha \subset \mathscr{T}$ the full subcategory of α-compact objects. Is every functor $H\colon (\mathscr{T}^\alpha)^{\mathrm{op}} \to \mathrm{Ab}$ that preserves products of $< \alpha$ objects and takes exact triangles to exact sequences of the form $H \cong \mathscr{T}(-,X)_{|\mathscr{T}^\alpha}$ for some X in $\mathscr{T}$? Is every natural transformation $\tau\colon \mathscr{T}(-,X)_{|\mathscr{T}^\alpha} \to \mathscr{T}(-,Y)_{|\mathscr{T}^\alpha}$ of the form $\tau = \mathscr{T}(-,f)_{|\mathscr{T}^\alpha}$ for some $f\colon X \to Y$ in $\mathscr{T}$? If the answer to both questions is positive we say that $\mathscr{T}$ satisfies α-Adams representability. A classical result going back to Brown and Adams shows that the stable homotopy category satisfies $\aleph_0$-Adams representability. The case $\alpha = \aleph_0$ is well understood thanks to the work of Christensen, Keller and Neeman. In this paper we develop an obstruction theory to decide when $\mathscr{T}$ satisfies α-Adams representability. We derive necessary and sufficient conditions of homological nature, and we compute several examples. In particular, we show that, for all $\alpha \geq \aleph_0$, there are rings whose derived category satisfies α-Adams representability and also rings for which the answer to the second question is no.

Contents

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Introduction

There are two classical representability theorems in the stable homotopy category $\mathscr{T}$. Any spectrum X gives rise to a cohomology theory $\mathscr{T}(-,X)\colon \mathscr{T}^{\mathrm{op}} \to \mathrm{Ab}$. The *Brown representability* theorem, [Bro62], says that any cohomology theory $H\colon \mathscr{T}^{\mathrm{op}} \to \mathrm{Ab}$ is of the form $H \cong \mathscr{T}(-,X)$ for some spectrum X. The *Adams representability* theorem, [Ada71], is a kind of analog for cohomology theories defined only on the full subcategory of compact spectra $\mathscr{T}^c \subset \mathscr{T}$. It asserts that any cohomology theory $H\colon (\mathscr{T}^c)^{\mathrm{op}} \to \mathrm{Ab}$ is of the form $H = \mathscr{T}(-,X)_{|\mathscr{T}^c}$ for some X, and, moreover, any natural transformation

$$\tau\colon \mathscr{T}(-,X)_{|\mathscr{T}^c} \longrightarrow \mathscr{T}(-,Y)_{|\mathscr{T}^c}$$

is induced by a map $f\colon X \to Y$, $\tau = \mathscr{T}(-,f)_{|\mathscr{T}^c}$. By Yoneda's lemma, the representing spectrum in Brown's theorem is unique and any natural transformation between cohomology theories on $\mathscr{T}$ comes from a unique map between the representing spectra. In Adams' theorem the spectrum X is still unique, but there may be different maps f representing a given natural transformation τ. Maps representing the trivial natural transformation are called *phantoms*. Brown proved Adams' theorem under the restrictive hypothesis that the cohomology theory H takes values in countable abelian groups. Adams' theorem allows to extend cohomology theories which are, in principle, only defined for compact spectra like topological K-theory defined in terms of vector bundles. Adams' theorem is stronger than Brown's, cf. [Ada71], and it also implies the representability of homology theories via the Spanier–Whitehead duality.

The analog of Brown's representability theorem is satisfied by a wide class of triangulated categories $\mathscr{T}$ including the *well generated* ones, i.e. if $\mathscr{T}$ is well generated any functor $H\colon \mathscr{T}^{\mathrm{op}} \to \mathrm{Ab}$ preserving products and taking exact triangles to exact sequences is of the form $H = \mathscr{T}(-,X)$ for some X in $\mathscr{T}$ [Nee01b, Theorem 8.3.3]. The simplest examples of well generated categories are the compactly generated ones. An object C in $\mathscr{T}$ is *compact* if the functor $\mathscr{T}(C,-)$ preserves direct sums, and $\mathscr{T}$ is *compactly generated* if it has coproducts and the full subcategory of compact objects $\mathscr{T}^c$ is essentially small and generates $\mathscr{T}$, i.e. an object X in $\mathscr{T}$ is trivial if and only if $\mathscr{T}(C,X) = 0$ for all C in $\mathscr{T}^c$. A compactly generated category $\mathscr{T}$ satisfies *Adams representability* if any additive functor $H\colon (\mathscr{T}^c)^{\mathrm{op}} \to \mathrm{Ab}$ taking exact triangles to exact sequences is of the form $H = \mathscr{T}(-,X)_{|\mathscr{T}^c}$ for some X in $\mathscr{T}$, and any natural transformation as τ above is induced by a map $f\colon X \to Y$,

$\tau = \mathscr{T}(-,f)_{|\mathscr{T}^c}$. Despite the category of compact objects contains much information about the whole category, Adams representability is seldom satisfied. It is satisfied, for instance, when $\mathscr{T}^c$ is essentially countable [Nee97]. This covers the stable homotopy category, but not the derived category $D(R)$ of a ring R unless R is countable. Adams representability is thoroughly studied in [Bel00] and [CKN01], with emphasis on derived categories of rings. It turns out to be strongly related to the pure global dimension of the ring R, a homological invariant connected to set theory, e.g. the first part of Adams representability for the derived category $D(\mathbb{C}\langle x,y\rangle)$ of a non-commutative polynomial ring on two variables over the complex numbers is equivalent to the continuum hypothesis.

Many well generated triangulated categories have not enough compact objects to generate, e.g. the homotopy category $K(\text{Proj-}R)$ of complexes of projective right R-modules over a ring R which is not right coherent [Nee08, Example 7.16]. There are even some well generated categories with no non-trivial compact objects at all, e.g. the derived category $D(\mathrm{Sh}/M)$ of sheaves of abelian groups on a connected non-compact paracompact manifold M of $\dim M \geq 1$ [Nee01a]. Therefore, in these contexts, Adams representability does not make much sense as considered above. In such cases, the role of compact objects is played by *α-compact objects* for a regular cardinal α. In a well generated category, for a large enough cardinal α, the category $\mathscr{T}^\alpha$ of α-compact objects is essentially small, closed under coproducts of $< \alpha$ objects, and generates $\mathscr{T}$. In this paper, we consider the following transfinite analog of Adams representability in $\mathscr{T}$.

Definition. Let α be a regular cardinal and $\mathscr{T}$ a well generated triangulated category. A functor $H\colon (\mathscr{T}^\alpha)^{\mathrm{op}} \to \mathrm{Ab}$ is *cohomological* if it takes exact triangles to exact sequences. We say that $\mathscr{T}$ satisfies *α-Adams representability* if the following two properties are satisfied:

- ARO_α Any cohomological functor $H\colon (\mathscr{T}^\alpha)^{\mathrm{op}} \to \mathrm{Ab}$ that preserves products of $< \alpha$ objects is isomorphic to $\mathscr{T}(-,X)_{|\mathscr{T}^\alpha}$ for some X in $\mathscr{T}$.
- ARM_α Any natural transformation $\tau\colon \mathscr{T}(-,X)_{|\mathscr{T}^\alpha} \to \mathscr{T}(-,Y)_{|\mathscr{T}^\alpha}$ is induced by a morphism $f\colon X \to Y$ in $\mathscr{T}$, $\tau = \mathscr{T}(-,f)_{|\mathscr{T}^\alpha}$.

The only case where these properties hold by obvious reasons for all α is the derived category $D(k)$ of a field k. Observe that if $\mathscr{T}$ is compactly generated $\aleph_0$-Adams representability is the same as Adams representability as considered above. Since $\mathrm{ARO}_{\aleph_0}$ and $\mathrm{ARM}_{\aleph_0}$ fail so often, it is also natural to consider ARO_α and ARM_α for $\alpha > \aleph_0$ in compactly generated categories.

For $\mathscr{T}$ a well generated triangulated category with models, Rosický stated in [Ros05] that ARO_α and ARM_α were satisfied for a proper class of regular cardinals α. Unfortunately, his proof contains a gap acknowledged in [Ros08] and [Ros09]. Nevertheless, this statement is a fairly natural question. Heuristically, since any well generated category is an increasing union of the subcategories of α-compact objects $\mathscr{T} = \cup_\alpha \mathscr{T}^\alpha$ by [Nee01b, Proposition 8.4.2], Brown representability can be regarded as the limit of ARO_α and ARM_α as α runs over all cardinals, and this question suggests that the limit statement is satisfied because it is satisfied in a 'cofinal' sequence.

Neeman obtained in [Nee09] striking consequences of Rosický's statement. One of them is that any covariant functor on a well generated triangulated category $H\colon \mathscr{T} \to \mathrm{Ab}$ preserving coproducts and taking exact triangles to exact sequences would be representable $H \cong \mathscr{T}(X,-)$. This is *Brown representability for the dual*

$\mathscr{T}^{\mathrm{op}}$. This result cannot be deduced from the Brown representability theorem for well generated categories since the opposite of a well generated category is never well generated. It was known for compactly generated triangulated categories, cf. [Nee98] and [Kra02], and it is a major open problem in the field for well generated categories.

In this paper, we show that some well generated triangulated categories do not satisfy α-Adams representability. For instance, we prove that $D(\mathbb{Z})$ satisfies ARM_α if and only if $\alpha = \aleph_0$. This uses the fact that the α-pure global dimension of $\mathbb{Z}$ is $\mathrm{pgd}_\alpha(\mathbb{Z}) > 1$ for $\alpha > \aleph_0$, cf. [BG12]. The *α-pure global dimension* of a ring R is the smallest n such that, for each right R-module M, there is a sequence

$$0 \to P_n \to \cdots \to P_1 \to M \to 0$$

where each P_i is a retract of a direct sum of right R-modules with $< \alpha$ generators and relations and

$$0 \to \mathrm{Hom}_R(Q, P_n) \to \cdots \to \mathrm{Hom}_R(Q, P_1) \to \mathrm{Hom}_R(Q, M) \to 0$$

is exact for any right R-module Q with $< \alpha$ generators and relations, cf. [JL89, Chapter 7].

A ring R is *α-coherent* if any right R-module with $< \alpha$ generators has a presentation with $< \alpha$ generators and relations. Rings of $\operatorname{card} R < \alpha$ are α-coherent, cf. [Mur11, Lemma 19]. We prove that, if R is α-coherent for some $\alpha > \aleph_0$ and $D(R)$ satisfies ARM_α, then $\mathrm{pgd}_\alpha(R) \leq 1$.

A ring R is *hereditary* if it has global dimension ≤ 1, e.g. $R = \mathbb{Z}$ and path algebras of quivers over a field. Hereditary rings are α-coherent for all $\alpha \geq \aleph_0$. For hereditary rings, we prove that ARO_α is equivalent to $\mathrm{pgd}_\alpha(R) \leq 2$ and that ARM_α is equivalent to $\mathrm{pgd}_\alpha(R) \leq 1$, $\alpha > \aleph_0$. The case $\alpha = \aleph_0$ was shown in [CKN01]. As we already mentioned, $\mathrm{pgd}_\alpha(\mathbb{Z}) > 1$ for all $\alpha > \aleph_0$, but nothing else is known about $\mathrm{pgd}_\alpha(\mathbb{Z})$ without set-theoretical assumptions. Under the continuum hypothesis, we prove that $\mathrm{pgd}_{\aleph_1}(\mathbb{Z}) = 2$, which implies $\mathrm{ARO}_{\aleph_1}$ for $D(\mathbb{Z})$, and more generally, if $2^{\aleph_{n-1}} = \aleph_n$, then $\mathrm{pgd}_{\aleph_n}(\mathbb{Z}) \leq n+1$. Computing $\mathrm{pgd}_\alpha(\mathbb{Z})$ becomes now a relevant problem since $\mathrm{pgd}_\alpha(\mathbb{Z}) = 2$ is equivalent to ARO_α for $D(\mathbb{Z})$. The first examples of rings with $\mathrm{pgd}_\alpha(R) > 1$ for all $\alpha \geq \aleph_0$ have been obtained in [BŠ13], e.g. $R = k[[x, y]]$ for k a field. As in the case of $\mathbb{Z}$, we do not know better bounds for $\mathrm{pgd}_\alpha(R)$ without set-theoretical hypotheses. These rings do not satisfy ARM_α for any $\alpha \geq \aleph_0$.

Concerning positive results, we show that the derived category $D(R)$ of a hereditary right pure-semisimple ring, e.g. the path algebra of a Dynkin quiver over a field, satisfies ARO_α and ARM_α for all α. Under the continuum hypothesis, we prove $\mathrm{ARO}_{\aleph_1}$ for the following categories, where R denotes a ring of $\operatorname{card} R \leq \aleph_1$: the stable homotopy category, the derived category $D(R)$ of right R-modules, the homotopy category $K(\text{Proj-}R)$ of complexes of projective right R-modules, the homotopy category $K(\text{Inj-}R)$ of complexes of injective right R-modules if R is right noetherian, the derived category $D(\mathrm{Sh}/M)$ of sheaves of abelian groups on a connected paracompact manifold, and the stable motivic homotopy category over a noetherian scheme of finite Krull dimension that can be covered by spectra of rings of cardinal $\leq \aleph_1$. We believe that set-theoretical assumptions are really necessary in these examples, as they are in order for $D(\mathbb{C}\langle x, y\rangle)$ to satisfy $\mathrm{ARO}_{\aleph_0}$. These results obtained under the continuum hypothesis suggest that for any *specific* cohomological functor $H\colon (\mathscr{T}^{\aleph_1})^{\mathrm{op}} \to \mathrm{Ab}$ preserving countable products there are many chances to find an object X with $H = \mathscr{T}(-,X)_{|_{\mathscr{T}^{\aleph_1}}}$, for if such an object did not exist the continuum hypothesis would be false.

We tackle ARO_α and ARM_α by means of a fairly general obstruction theory for triangulated categories. We consider a well generated triangulated category $\mathscr{T}$ and a full subcategory $\mathscr{C} \subset \mathscr{T}^\alpha$ closed under (de)suspensions and coproducts of $< \alpha$ objects which generates $\mathscr{T}$. We do not require $\mathscr{C}$ to be triangulated, although in this paper the main example is $\mathscr{C} = \mathscr{T}^\alpha$. We consider the *restricted Yoneda functor*,

$$S_\alpha \colon \mathscr{T} \longrightarrow \mathrm{Mod}_\alpha(\mathscr{C}), \quad S_\alpha(X) = \mathscr{T}(-,X)_{|_{\mathscr{C}}},$$

where $\mathrm{Mod}_\alpha(\mathscr{C})$ is the abelian category of *α-continuous (right) $\mathscr{C}$-modules*, i.e. functors $\mathscr{C}^{\mathrm{op}} \to \mathrm{Ab}$ preserving products of $< \alpha$ objects. Morphisms in the kernel of S_α are called *phantom maps*. We interpolate the functor S_α by an inverse sequence of categories

$$\mathscr{T} \to \cdots \to \mathbf{Post}^{\simeq}_{n+1} \xrightarrow{t_n} \mathbf{Post}^{\simeq}_{n} \to \cdots \to \mathbf{Post}^{\simeq}_{0} \xrightarrow{\sim} \mathrm{Mod}_\alpha(\mathscr{C}).$$

For each step $t_n \colon \mathbf{Post}^{\simeq}_{n+1} \to \mathbf{Post}^{\simeq}_{n}$, we define obstructions to the lifting of objects and morphisms along t_n. Obstructions take values in Ext groups in $\mathrm{Mod}_\alpha(\mathscr{C})$. The obstructions for the lifting of objects were first considered in [BKS04] for $\alpha = \aleph_0$. In addition, we prove that the induced functor

$$t \colon \mathscr{T} \longrightarrow \mathbf{Post}^{\simeq}_{\infty} = \lim_n \mathbf{Post}^{\simeq}_{n}$$

is full and essentially surjective. We also analyze the kernel of t_n and, moreover, we show that the kernel of the functor t is the ideal of *∞-phantom maps*, i.e. maps $f \colon X \to Y$ in $\mathscr{T}$ which decompose as a product $f = f_n \cdots f_1$ of n phantom maps f_i, $1 \le i \le n$, for all $n \ge 1$. Furthermore, we prove that ∞-phantom maps form a square-zero ideal, i.e. the composition of two ∞-phantom maps is always zero. This is a new result even for a compactly generated triangulated category $\mathscr{T}$ and $\mathscr{C} = \mathscr{T}^c$.

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1. The restricted Yoneda functor

For the basic notions and properties of triangulated categories used in this section we refer the reader to [Nee01b], [Kra00] and [Kra10].

Throughout this paper α is a regular cardinal, $\mathscr{T}$ is a well generated triangulated category with suspension functor Σ, and $\mathscr{C} \subset \mathscr{T}^\alpha$ is an essentially small full subcategory such that:

(1) it is closed under (de)suspensions,
(2) it has coproducts of less than α objects, and
(3) it generates $\mathscr{T}$, i.e. an object X in $\mathscr{T}$ is zero if and only if $\mathscr{T}(C,X) = 0$ for all C in $\mathscr{C}$.

In particular, $\mathscr{T}$ is α-compactly generated in the sense of Neeman [Nee01b] and has products and coproducts. We do not require $\mathscr{C}$ to be triangulated. If it were, then necessarily $\mathscr{C} = \mathscr{T}^{\alpha}$. In order to avoid absurd situations, we assume that both $\mathscr{C}$ and $\mathscr{T}$ are non-trivial, i.e. $\mathscr{C}$ contains at least one object $X \neq 0$.

Let $\mathrm{Mod}_{\alpha}(\mathscr{C})$ be the abelian category of functors $\mathscr{C}^{\mathrm{op}} \to \mathrm{Ab}$ preserving products of less than α objects. Such functors are called *α-continuous (right) $\mathscr{C}$-modules*. This category is locally α-presentable, α-filtered colimits are exact, and representable functors form a set of α-presentable projective generators. Moreover, α-filtered colimits in $\mathrm{Mod}_{\alpha}(\mathscr{C})$ are computed pointwise, i.e. if Λ is an α-filtering category, $\Lambda \to \mathrm{Mod}_{\alpha}(\mathscr{C})\colon \lambda \mapsto F_{\lambda}$ is a diagram of α-continuous $\mathscr{C}$-modules, and C is an object in $\mathscr{C}$, then

$$(\operatorname*{colim}_{\lambda\in\Lambda} F_{\lambda})(C) = \operatorname*{colim}_{\lambda\in\Lambda}(F_{\lambda}(C)),$$

where the first colimit is taken in $\mathrm{Mod}_{\alpha}(\mathscr{C})$ and the second one is in the category Ab of abelian groups.

The *restricted Yoneda functor*,

$$S_{\alpha}\colon \mathscr{T} \longrightarrow \mathrm{Mod}_{\alpha}(\mathscr{C}), \quad S_{\alpha}(X) = \mathscr{T}(-,X)_{|\mathscr{C}},$$

preserves products and coproducts, takes exact triangles to exact sequences, and reflects isomorphisms. If $\mathrm{Add}\,(\mathscr{C}) \subset \mathscr{T}$ denotes the smallest subcategory closed under coproducts and retracts containing $\mathscr{C}$, then S_{α} induces an equivalence between $\mathrm{Add}\,(\mathscr{C})$ and the full subcategory of projective objets in $\mathrm{Mod}_{\alpha}(\mathscr{C})$. Moreover, if P is in $\mathrm{Add}\,(\mathscr{C})$ and X is in $\mathscr{T}$, then S_{α} induces an isomorphism

$$\mathscr{T}(P,X) \cong \mathrm{Hom}_{\alpha,\mathscr{C}}(S_{\alpha}(P), S_{\alpha}(X)),$$

where $\mathrm{Hom}_{\alpha,\mathscr{C}}$ denotes the morphism sets in $\mathrm{Mod}_{\alpha}(\mathscr{C})$.

Notice that properties ARO_{α} and ARM_{α}, defined in the introduction, translate as follows for $\mathscr{C} = \mathscr{T}^{\alpha}$:

ARO_{α} The essential image of S_{α} is the class of cohomological functors in $\mathrm{Mod}_{\alpha}(\mathscr{C})$.
ARM_{α} The functor S_{α} is full.

Denote $\mathrm{pd}(A)$ the projective dimension of an object A in an abelian category $\mathscr{A}$.

Proposition 1.1. *If S_{α} is full, then $\mathrm{pd}(S_{\alpha}(X)) \leq 1$ for all X in $\mathscr{T}$.*

The proof of this proposition is essentially the same as the proof of [Nee97, Lemma 4.1]. We will use the following elementary lemma.

Lemma 1.2. *If $X \stackrel{f}{\to} Y \to Z \to \Sigma X$ is an exact triangle and f decomposes as $f = \binom{f'}{0}\colon X \to Y' \oplus Y'' = Y$, then this exact triangle is the direct sum of an exact triangle*

$$X \xrightarrow{f'} Y' \longrightarrow Z' \longrightarrow \Sigma X$$

and $0 \to Y'' \stackrel{1}{\to} Y'' \to 0$. In particular $Z \cong Z' \oplus Y''$.

Proof of Proposition 1.1. Choose a projective presentation of $S_{\alpha}(X)$,

$$S_{\alpha}(P_1) \longrightarrow S_{\alpha}(P_0) \twoheadrightarrow S_{\alpha}(X).$$

It comes from unique morphisms $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} X$ with $p_0p_1 = 0$, therefore p_0 factors through the mapping cone of p_1 in an exact triangle

$$\begin{tikzcd} P_1 \arrow[r, "p_1"] & P_0 \arrow[r, "i"] \arrow[d, "p_0"'] & Y \arrow[r, "q"] \arrow[dl, "p'"] & \Sigma P_1 \\ & X. & & \end{tikzcd}$$

The universal property of a cokernel shows that $S_\alpha(i)$ factors through $S_\alpha(p_0)$,

$$\begin{tikzcd} S_\alpha(P_1) \arrow[r, "S_\alpha(p_1)"] & S_\alpha(P_0) \arrow[r, "S_\alpha(i)"] \arrow[d, two heads, "S_\alpha(p_0)"'] & S_\alpha(Y) \arrow[r, "S_\alpha(q)"] & S_\alpha(\Sigma P_1) \\ & S_\alpha(X). \arrow[ur, hook, "\phi"'] & & \end{tikzcd}$$

Since $S_\alpha(p_0)$ is an epimorphism and

$$S_\alpha(p')\phi S_\alpha(p_0) = S_\alpha(p')S_\alpha(i) = S_\alpha(p'i) = S_\alpha(p_0),$$

we deduce that $S_\alpha(p')\phi = 1_{S_\alpha(X)}$. Using that the functor S_α is full, we can take a morphism $i' \colon X \to Y$ with $\phi = S_\alpha(i')$. Hence, $S_\alpha(p')\phi = S_\alpha(p'i') = 1_{S_\alpha(X)}$ and, since S_α reflects isomorphisms, $p'i'$ is an automorphism of X, so Y decomposes as $(i', i'') \colon X \oplus Z \cong Y$ for some Z and i''. On the other hand, since the morphism $S_\alpha(i)$ factors as $S_\alpha(i')S_\alpha(p)$ and $S_\alpha(P_0)$ is projective, i itself factors as $i = i'p_0$, i.e. i decomposes as $i = \binom{p_0}{0} \colon P_0 \to X \oplus Z \cong Y$. Now, Lemma 1.2 shows that $P_1 \cong P_1' \oplus \Sigma^{-1}Z$ and that there is an exact triangle

$$P_1' \longrightarrow P_0 \xrightarrow{p_0} X \longrightarrow \Sigma P_1'.$$

In particular, $S_\alpha(P_1')$ is projective. Since $S_\alpha(p_0)$ is an epimorphism, the image under S_α of the previous exact triangle produces a length 1 projective resolution of $S_\alpha(X)$,

$$S_\alpha(P_1') \hookrightarrow S_\alpha(P_0) \twoheadrightarrow S_\alpha(X).$$

□

We derive the following necessary condition for ARM_α.

Corollary 1.3. *If $\mathscr{T}$ satisfies ARM_α, then $\mathrm{pd}(S_\alpha(X)) \leq 1$ for all X in $\mathscr{T}$.*

2. An obstruction theory for the restricted Yoneda functor

In this section, we describe the formal properties of the obstruction theory developed in Section 6. We derive a sufficient condition for ARO_α (Corollary 2.13) and necessary and sufficient conditions for ARM_α (Corollary 2.5) and for the α-Adams representability theorem (Corollary 2.15).

The following notion of exact sequence of categories generalizes [Bau89, Definition IV.4.10] by incorporating an obstruction κ to the lifting of objects.

Definition 2.1. Given an additive category $\mathscr{B}$, a *$\mathscr{B}$-bimodule M* is a biadditive functor $M \colon \mathscr{B}^{\mathrm{op}} \times \mathscr{B} \to \mathrm{Ab}$. The canonical example is the bimodule defined by morphism sets, that we denote $\mathscr{B} = \mathscr{B}(-,-)$. As usual, we can change coefficients along additive functors $\mathscr{A} \to \mathscr{B}$, so $\mathscr{B}$-bimodules become $\mathscr{A}$-bimodules.

An *exact sequence of categories*

$$\begin{array}{ccccccc} & & & & M_0 & & \\ & & & & \uparrow{\scriptstyle\kappa} & & \\ M_2 & \xrightarrow{\imath} & \mathscr{A} & \xrightarrow{t} & \mathscr{B} & \xrightarrow{\theta} & M_1 \end{array}$$

consists of an additive functor t, three $\mathscr{B}$-bimodules M_i, $i = 0, 1, 2$, an exact sequence

$$M_2(t(X), t(Y)) \xrightarrow{\imath_{X,Y}} \mathscr{A}(X, Y) \xrightarrow{t} \mathscr{B}(t(X), t(Y)) \xrightarrow{\theta_{X,Y}} M_1(t(X), t(Y))$$

for any two objects X and Y in $\mathscr{A}$, and an element

$$\kappa(B) \in M_0(B, B)$$

for any object B in $\mathscr{B}$. The following conditions must be satisfied:

(1) For any morphism $f\colon B \to C$ in $\mathscr{B}$,
$$f \cdot \kappa(B) = \kappa(C) \cdot f \in M_0(B, C).$$
(2) $\kappa(B) = 0$ if and only if there exists an object A in $\mathscr{A}$ with $t(A) = B$.
(3) Given objects X, Y, Z in $\mathscr{A}$ and morphisms $t(X) \xrightarrow{f} t(Y) \xrightarrow{g} t(Z)$ in $\mathscr{B}$,
$$\theta_{X,Z}(gf) = \theta_{Y,Z}(g) \cdot f + g \cdot \theta_{X,Y}(f) \in M_1(X, Z).$$
(4) For any object X in $\mathscr{A}$ and any $e \in M_1(t(X), t(X))$ there exists an object $X' = X + e$ in $\mathscr{A}$ with $t(X) = t(X')$ and $\theta_{X,X'}(\mathrm{id}_{t(X)}) = e$.
(5) $\imath$ is a morphism of $\mathscr{A}$-bimodules.

We sometimes omit the subscripts from $\imath$ and θ so as not to overload notation.

In an exact sequence of categories, κ is a 0-dimensional element in Baues–Wirsching cohomology of categories $H^0(\mathscr{B}, M_0)$, cf. [BW85]. Moreover, the rest of the exact sequence is determined by a 1-dimensional and a 2-dimensional cohomology class, compare [Bau89, Chapter IV].

A triangulated category $\mathscr{T}$ is regarded as a graded category with graded morphism sets

$$\mathscr{T}^*(X, Y) = \bigoplus_{n \in \mathbb{Z}} \mathscr{T}(X, \Sigma^n Y).$$

Since $\mathscr{C}$ is closed under (de)suspensions, Σ admits an essentially unique exact extension to $\mathrm{Mod}_\alpha(\mathscr{C})$ compatible with the restricted Yoneda functor, i.e. the following diagram commutes up to natural isomorphism:

$$\begin{array}{ccc} \mathscr{T} & \xrightarrow[\sim]{\Sigma} & \mathscr{T} \\ {\scriptstyle S_\alpha}\downarrow & & \downarrow{\scriptstyle S_\alpha} \\ \mathrm{Mod}_\alpha(\mathscr{C}) & \xrightarrow[\sim]{\Sigma} & \mathrm{Mod}_\alpha(\mathscr{C}). \end{array}$$

The functor Σ endows $\mathrm{Mod}_\alpha(\mathscr{C})$ with the structure of a graded abelian category. Graded morphism sets in $\mathrm{Mod}_\alpha(\mathscr{C})$ are defined as in $\mathscr{T}$,

$$\mathrm{Hom}^*_{\alpha,\mathscr{C}}(M, N) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\alpha,\mathscr{C}}(M, \Sigma^n N).$$

In a graded abelian category we also have graded Ext functors that we denote $\mathrm{Ext}^{p,q}_{\alpha,\mathscr{C}}$, where p indicates the length of the extension, i.e. the p^{th} derived functor of $\mathrm{Hom}_{\alpha,\mathscr{C}}$, and q is the internal degree coming from the graded $\mathrm{Hom}^*_{\alpha,\mathscr{C}}$. Notice that $\mathrm{Ext}^{p,q}_{\alpha,\mathscr{C}}$ is a $\mathrm{Mod}_\alpha(\mathscr{C})$-bimodule. We refer to [Str68] for additive and abelian category theory in the graded setting.

The following theorem summarizes the main results of Section 6.

Theorem 2.2. *There is a sequence of exact sequences of categories, $n \geq 0$,*

$$\begin{tikzcd} & & \mathrm{Ext}^{n+3,-1-n}_{\alpha,\mathscr{C}} & \\ \mathrm{Ext}^{n+1,-1-n}_{\alpha,\mathscr{C}} \arrow[r, "\imath_n"] & \mathbf{Post}^{\simeq}_{n+1} \arrow[r, "t_n"] & \mathbf{Post}^{\simeq}_{n} \arrow[u, "\kappa_n"] \arrow[r, "\theta_n"] & \mathrm{Ext}^{n+2,-1-n}_{\alpha,\mathscr{C}} \end{tikzcd}$$

with $\mathbf{Post}^{\simeq}_0 \simeq \mathrm{Mod}_\alpha(\mathscr{C})$ and a full and essentially surjective functor

$$\mathscr{T} \longrightarrow \mathbf{Post}^{\simeq}_\infty = \lim_n \mathbf{Post}^{\simeq}_n$$

such that the composite $\mathscr{T} \to \mathbf{Post}^{\simeq}_\infty \to \mathbf{Post}^{\simeq}_0 \simeq \mathrm{Mod}_\alpha(\mathscr{C})$ is naturally isomorphic to the restricted Yoneda functor S_α.

In Section 6 we omit the subscript n from $\imath$, θ and κ. Under the hypotheses of the following corollary all obstructions vanish since the recipient bimodules vanish.

Corollary 2.3. *Under the standing assumptions:*

1. *If F is an α-continuous $\mathscr{C}$-module with $\mathrm{pd}(F) \leq 2$, then $F \cong S_\alpha(X)$ for some X in $\mathscr{T}$.*
2. *If $\mathrm{pd}(S_\alpha(X)) \leq 1$, then any morphism $\tau\colon S_\alpha(X) \to S_\alpha(Y)$ is $\tau = S_\alpha(f)$ for some $f\colon X \to Y$ in $\mathscr{T}$.*

Combining Corollary 2.3 with Proposition 1.1 we obtain the following results.

Corollary 2.4. *The functor S_α is full if and only if its essential image consists of the α-continuous $\mathscr{C}$-modules F with $\mathrm{pd}(F) \leq 1$.*

Corollary 2.5. *The category $\mathscr{T}$ satisfies ARM_α if and only if $\mathrm{pd}(S_\alpha(X)) \leq 1$ for all X in $\mathscr{T}$.*

Remark 2.6. A different approach to the lifting of morphisms along the restricted Yonead functor for $\alpha = \aleph_0$ is developed in [BK03].

We now list some examples of $\mathscr{T}$ and $\mathscr{C} \neq \mathscr{T}^\alpha$ were it would be interesting to apply the obstruction theory summarized in Theorem 2.2. In all cases $\alpha = \aleph_0$:

- $\mathscr{T}$ the stable module category of the group ring kG of a finite group G over a field k and $\mathscr{C}$ the full subcategory of finite-dimensional k-vector spaces with the trivial action of G. In this case, $\mathrm{Mod}_{\aleph_0}(\mathscr{C})$ is equivalent to the category of $H^*(G,k)$-modules, where $H^*(G,k)$ is the Tate cohomology ring, and the restricted Yoneda functor identifies with $M \mapsto H^*(G,M)$.
- $\mathscr{T}$ the homotopy category of modules over a ring spectrum R and $\mathscr{C}$ the full subcategory spanned by free R-modules, i.e. finite coproducts of suspensions of R. In this case, $\mathrm{Mod}_{\aleph_0}(\mathscr{C})$ is the category of $\pi_*(R)$-modules and the restricted Yoneda functor corresponds to $M \mapsto \pi_*(M)$.
- $\mathscr{T}$ the derived category of a differential graded algebra A and $\mathscr{C}$ the full subcategory of free A-modules, i.e. finite coproducts of shifts of A. Here $\mathrm{Mod}_{\aleph_0}(\mathscr{C})$ is the category of $H_*(A)$-modules and the restricted Yoneda functor identifies with $M \mapsto H_*(M)$

The first obstruction κ_0 to the realizability of an object has been considered with detail in these three cases, see [BKS04, Sag08, GH08] respectively. Indeed, [BKS04] is where the obstructions κ_n to the realizability of objects were first treated systematically.

We now consider α-flat objects and their connection with α-Adams representability.

Definition 2.7. Let $\mathscr{A}$ be a cocomplete abelian category. An *α-flat object* A in $\mathscr{A}$ is an α-filtered colimit of α-presentable projective objects $A = \operatorname{colim}_{\lambda\in\Lambda} P_\lambda$. The *$\alpha$-flat global dimension* of $\mathscr{A}$ is

$$\operatorname{fgd}_\alpha(\mathscr{A}) = \sup\{\operatorname{pd}(A) \mid A \text{ is } \alpha\text{-flat}\}.$$

Remark 2.8. An α-flat object $A = \operatorname{colim}_{\lambda\in\Lambda} P_\lambda$ has a canonical projective resolution of the form

$$\cdots \to \bigoplus_{\lambda\to\mu\to\nu\in\Lambda} P_\lambda \longrightarrow \bigoplus_{\lambda\to\mu\in\Lambda} P_\lambda \longrightarrow \bigoplus_{\lambda\in\Lambda} P_\lambda \twoheadrightarrow A.$$

These direct sums are indexed by the simplices of the nerve $N\Lambda$ of the category Λ indexing the colimit, i.e. for each n, $N_n\Lambda = \{\text{chains of } n \text{ composable maps in } \Lambda\}$. In particular, for any other object B in $\mathscr{A}$ the higher Ext's

$$\operatorname{Ext}^n_{\mathscr{A}}(A,B) = \lim_{\lambda\in\Lambda}{}^n \operatorname{Hom}_{\mathscr{A}}(P_\lambda, B)$$

are the derived functors of the inverse limit.

Remark 2.9. In $\operatorname{Mod}_\alpha(\mathscr{T}^\alpha)$, the α-flat objects coincide with the cohomological functors, cf. [Nee01b, Section 7.2].

The α-flat global dimension of $\mathscr{C}$ can be bounded above if the cardinal of $\mathscr{C}$ is not too large.

Definition 2.10. The *cardinal* of a small category $\mathscr{C}$ is

$$\operatorname{card}\mathscr{C} = \operatorname{card} \coprod_{x,y\in S} \mathscr{C}(x,y),$$

where S is a set of isomorphism classes of objects in $\mathscr{C}$.

Lemma 2.11. *If $\mathscr{C}$ is a non-trivial additive category with coproducts of less than α objects, then* $\operatorname{card}\mathscr{C} \geq \alpha$.

Proof. If $X \neq 0$ the identity in X is non-trivial, so $\operatorname{card}\mathscr{C}(X,X) \geq 2$. For $\beta < \alpha$,

$$\mathscr{C}(\coprod_\beta X, X) = \prod_\beta \mathscr{C}(X,X), \qquad \operatorname{card}\prod_\beta \mathscr{C}(X,X) \geq 2^\beta.$$

Hence, $\operatorname{card}\mathscr{C} \geq \sup_{\beta<\alpha} 2^\beta$. We now distinguish two cases, if $\alpha = \gamma^+$ is a successor, then $\sup_{\beta<\alpha} 2^\beta = 2^\gamma \geq \gamma^+ = \alpha$, and, if α is a limit cardinal, then $\sup_{\beta<\alpha} 2^\beta \geq \sup_{\beta<\alpha} \beta = \alpha$. $\square$

In Section 8 we show, under the generalized continuum hypothesis, that there is always a large enough cardinal α such that $\operatorname{card}\mathscr{T}^\alpha = \alpha$.

By Lemma 2.11, the hypothesis of the following proposition can only be satisfied if $\alpha \leq \aleph_n$.

Proposition 2.12. *If* $\operatorname{card}\mathscr{C} \leq \aleph_n$*, then* $\operatorname{fgd}_\alpha(\operatorname{Mod}_\alpha(\mathscr{C})) \leq n+1$.

Proof. The full inclusion $\mathrm{Mod}_\alpha(\mathscr{C}) \subset \mathrm{Mod}_{\aleph_0}(\mathscr{C})$ preserves α-filtered colimits. The α-presentable projective objects in $\mathrm{Mod}_\alpha(\mathscr{C})$ are the retracts of the representable functors $\mathscr{C}(-,X)$, which also coincide with the $\aleph_0$-presentable objects in $\mathrm{Mod}_{\aleph_0}(\mathscr{C})$. Therefore α-flat objects in $\mathrm{Mod}_\alpha(\mathscr{C})$ are also α-flat in $\mathrm{Mod}_{\aleph_0}(\mathscr{C})$, in particular $\aleph_0$-flat. Moreover, by Remark 2.8, if F is an α-continuous $\mathscr{C}$-module and $H = \mathrm{colim}_{\lambda\in\Lambda} P_\lambda$ is an α-flat α-continuous $\mathscr{C}$-module, then

$$\mathrm{Ext}^n_{\alpha,\mathscr{C}}(H,F) = \lim_{\lambda\in\Lambda}{}^n \mathrm{Hom}_{\alpha,\mathscr{C}}(P_\lambda,F) = \lim_{\lambda\in\Lambda}{}^n \mathrm{Hom}_{\aleph_0,\mathscr{C}}(P_\lambda,F) = \mathrm{Ext}^n_{\aleph_0,\mathscr{C}}(H,F).$$

This is proven in [Nee01b, Proposition 7.5.5] assuming that $\mathscr{C}$ is triangulated, but this hypothesis is not really used. If $\mathrm{card}\,\mathscr{C} \leq \aleph_n$, then any $\aleph_0$-flat $\aleph_0$-continuous $\mathscr{C}$-module has projective dimension $\leq n+1$ in $\mathrm{Mod}_{\aleph_0}(\mathscr{C})$, see [Sim77, Corollary 3.13]. Hence the proposition follows from the previous equation. □

We now concentrate in the case $\mathscr{C} = \mathscr{T}^\alpha$. The following sufficient condition for ARO_α follows from Corollary 2.3 and Remark 2.9.

Corollary 2.13. *If* $\mathrm{fgd}_\alpha(\mathrm{Mod}_\alpha(\mathscr{T}^\alpha)) \leq 2$ *, then* $\mathscr{T}$ *satisfies* ARO_α*.*

For the following corollary we also use Proposition 2.12. The restrictions on the cardinal α are imposed by Lemma 2.11.

Corollary 2.14. *Let* α *be* $\aleph_0$ *or* $\aleph_1$*. If* $\mathrm{card}\,\mathscr{T}^\alpha \leq \aleph_1$*, then* $\mathscr{T}$ *satisfies* ARO_α*.*

The following homological characterization of α-Adams representability is a consequence of Corollaries 2.4 and 2.13 and Remark 2.9.

Corollary 2.15. *A triangulated category* $\mathscr{T}$ *satisfies* α*-Adams representability if and only if* $\mathrm{fgd}_\alpha(\mathrm{Mod}_\alpha(\mathscr{T}^\alpha)) \leq 1$.

Using Proposition 2.12, we obtain Neeman's sufficient condition for $\aleph_0$-Adams representability, cf. [Nee97].

Corollary 2.16. *If* $\mathrm{card}\,\mathscr{T}^{\aleph_0} \leq \aleph_0$*, then* $\mathscr{T}$ *satisfies* $\aleph_0$*-Adams representability.*

Remark 2.17. In the case $\alpha = \aleph_0$, Beligiannis proves in [Bel00, Theorem 11.8] that $\mathscr{T}$ satisfies $\mathrm{ARM}_{\aleph_0}$ if and only if $\mathrm{fgd}_{\aleph_0}(\mathrm{Mod}_{\aleph_0}(\mathscr{T}^{\aleph_0})) \leq 1$. Thus, by Corollary 2.15, $\mathrm{ARM}_{\aleph_0}$ implies $\mathrm{ARO}_{\aleph_0}$.

A crucial step in his proof if that, since $\mathrm{Mod}_{\aleph_0}(\mathscr{T}^{\aleph_0})$ is a Grothendieck category, it follows from [Sim77, Theorem 2.7] that $\mathrm{fgd}_{\aleph_0}(\mathrm{Mod}_{\aleph_0}(\mathscr{T}^{\aleph_0})) = \sup\{\mathrm{pd}(A) \mid A \text{ is } \alpha\text{-flat and } \mathrm{pd}(A) < \infty\}$.

The fact that $\mathrm{Mod}_{\aleph_0}(\mathscr{T}^{\aleph_0})$ is Grothendieck is used in order to apply (in each step of an inductive argument) the Auslander Lemma: If $X = \bigcup_{i\in I} X_i$, where $\{X_i\}_{i\in I}$ well ordered by inclusion, and $\mathrm{pd}(X_{i+1}/X_i) \leq k$, then $\mathrm{pd}(X) \leq k$. However, for $\alpha > \aleph_0$, $\mathrm{Mod}_\alpha(\mathscr{T}^\alpha)$ need not be a Grothendieck category because filtered colimits need not be exact, only α-filtered colimits are exact. In fact, $\mathrm{Mod}_\alpha(\mathscr{T}^\alpha)$ can fail to have enough injectives, cf. [Nee01b, Section C.4]. The authors have proved an analog of the Auslander Lemma that applies to $\mathrm{Mod}_{\aleph_n}(\mathscr{T}^{\aleph_n})$ (to be published elsewhere). However, it does not help in extending Beligiannis' result for larger cardinals, since, in the analog hypotheses of the Auslander Lemma above, we obtain $\mathrm{pd}(X) \leq k+n$, which hampers the inductive argument.

Using a completely different approach, we will extend Beligiannis' result to the case $\mathscr{T} = D(R)$ for a hereditary ring R and any α, see Theorem 3.3 and Corollary 3.16.

3. Transfinite Adams representability in the derived category of a ring

In this section we consider ARO_α and ARM_α for the derived category $D(R)$ of an α-coherent ring R. The main result is Theorem 3.3, which gives a necessary condition for ARM_α, and also necessary and sufficient conditions for both ARO_α and ARM_α if R is hereditary. We also prove $\mathrm{ARO}_{\aleph_1}$ for rings of cardinality $\leq \aleph_1$ under the continuum hypothesis (Proposition 3.7). All modules considered in this section are right modules.

Definition 3.1. Let R be a ring and α a regular cardinal. An R-module is *α-generated* if it has a set of generators of cardinal $< \alpha$, it is *α-presentable* if it is the quotient of two α-generated modules. The ring R is *α-coherent* if all α-generated modules are α-presentable. It is enough to check this condition on ideals, cf. [JL89, Chapter 7].

Remark 3.2. Alternatively, an R-module P is α-presentable if it admits a free presentation

$$\bigoplus_J R \longrightarrow \bigoplus_I R \twoheadrightarrow P$$

with $\operatorname{card} I, \operatorname{card} J < \alpha$. Any α-presentable R-module is α-generated. The converse is true for projective R-modules.

If $\operatorname{card} R < \alpha$, then R is α-coherent, cf. [Mur11, Lemma 19]. Moreover, hereditary rings are α-coherent for all α since ideals are projective.

We now state the main result of this section. We make use of the α-pure global dimension of a ring $\mathrm{pgd}_\alpha(R)$ as it was defined in the introduction, cf. [JL89, Chapter 7], although below we give a more general definition for abelian categories.

Theorem 3.3. *Let R be an α-coherent ring, $\alpha > \aleph_0$. If $D(R)$ satisfies ARM_α, then $\mathrm{pgd}_\alpha(R) \leq 1$. Moreover, if R is hereditary, then*

1. ARO_α *for* $D(R) \Leftrightarrow \mathrm{pgd}_\alpha(R) \leq 2$, *and*
2. ARM_α *for* $D(R) \Leftrightarrow \mathrm{pgd}_\alpha(R) \leq 1$.

We prove Theorem 3.3 at the end of this section. The version for $\alpha = \aleph_0$, proved in [CKN01, Theorem 2.13], also requires that finitely presented R-modules have finite projective dimension, which is of course true for R hereditary.

Example 3.4. A consequence of Theorem 3.3 is that ARM_α is not satisfied for the derived category of α-coherent rings R such that $\mathrm{pgd}_\alpha(R) > 1$. Hence we can use computations of lower bounds to α-pure projective dimensions in [BL82], [BG12], and [BŠ13] to show that ARM_α is not satisfied for rings R and regular cardinals α as indicated.

1. $R = \mathbb{Z}$ for $\alpha > \aleph_0$.
2. Let k be an uncountable field and α any regular cardinal or k a countable field and $\alpha > \aleph_0$.
   - (a) $R = k[x, y]$.
   - (b) R the path algebra of a finite quiver without oriented cycles which is not a Dynkin quiver.
3. $R = k[[x, y]]$ for any field k and any regular cardinal α.

Remark 3.5. It is well known that a ring R has $\operatorname{pgd}_\alpha(R) = 0$ for some α if and only if $\operatorname{pgd}_{\aleph_0}(R) = 0$, see [JL89, Theorem 8.4]. These rings are called *(right) pure-semisimple*. Rings of finite representation type are (two-sided) pure-semisimple [JL89, Theorem 8.8]. If R is hereditary and pure-semisimple, e.g. the path algebra of a Dynkin quiver, then $D(R)$ satisifes ARM_α for all $\alpha \geq \aleph_0$. So far we do not know of any ring R with $\operatorname{pgd}_\alpha(R) = 1$ for some $\alpha > \aleph_0$.

Remark 3.6. The most important open problem concerning α-Adams representability for derived categories of rings is the following:

- Is there any ring R and any $\alpha > \aleph_0$ for which $D(R)$ does *not* satisfy ARO_α?

Our feeling is that the situation should be similar to ARM_α, i.e. there should be rings which do not satisfy ARO_α for any $\alpha > \aleph_0$, even for any α. By Theorem 3.3, it would be enough to find a hereditary ring with $\operatorname{pgd}_\alpha(R) > 2$. So far, there are no known computations of $\operatorname{pgd}_\alpha(R)$ for uncountable α, except from what is mentioned in Example 3.4 and in the Remark 3.5. We now obtain upper bounds under the (generalized) continuum hypothesis.

The following result proves $\mathrm{ARO}_{\aleph_1}$ for rings of cardinality $\leq \aleph_1$ under the continuum hypothesis. The proof is given after some preliminary considerations.

Proposition 3.7. *Let α be an inaccessible cardinal or $\alpha = \beta^+ = 2^\beta$. If R is a ring of* $\operatorname{card} R \leq \alpha$, *then* $\operatorname{card} D(R)^\alpha \leq \alpha$. *In particular, if* $\operatorname{card} R \leq \aleph_n = 2^{\aleph_{n-1}}$ *then* $\operatorname{pgd}_{\aleph_n}(R) \leq n$. *Moreover, if* $\operatorname{card} R \leq \aleph_1$ *and the continuum hypothesis holds then* $\operatorname{pgd}_{\aleph_1}(R) \leq 1$ *and $D(R)$ satisfies* $\mathrm{ARO}_{\aleph_1}$.

Remark 3.8. Recall from Example 3.4 that for the following rings R, $\operatorname{pgd}_{\aleph_1}(R) > 1$.

(1) $R = \mathbb{Z}$.
(2) Let k be a field of $\operatorname{card} k \leq \aleph_1$.
    (a) $R = k[x, y]$.
    (b) R the path algebra of a finite quiver without oriented cycles which is not a Dynkin quiver.
(3) $R = k[[x, y]]$ for a countable field k and any regular cardinal α.

The las part of Proposition 3.7 applies to these rings, therefore, under the continuum hypothesis, $\operatorname{pgd}_{\aleph_1}(R) = 2$ and $D(R)$ satisfies $\mathrm{ARO}_{\aleph_1}$.

Moreover, $1 < \operatorname{pgd}_{\aleph_n}(R) \leq n + 1$ if $2^{\aleph_{n-1}} = \aleph_n$, but the explicit computation of $\operatorname{pgd}_\alpha(R)$ for $\alpha > \aleph_1$ remains an open problem. It is not known if there is a lower bound better than > 1 or if one can compute $\operatorname{pgd}_{\aleph_n}(R)$ without assumptions related to the (generalized) continuum hypothesis.

Definition 3.9. Let α be a regular cardinal and $\mathscr{A}$ a locally α-presentable abelian category with exact α-filtered colimits and a set of α-presentable projective generators. A short exact sequence $A \hookrightarrow B \twoheadrightarrow C$ is *α-pure* if

$$\mathscr{A}(P, A) \hookrightarrow \mathscr{A}(P, B) \twoheadrightarrow \mathscr{A}(P, C)$$

is shot exact for any α-presentable object P, or equivalently, if it is an α-filtered colimit of split short exact sequences.

A sequence $\cdots \to A_{n+1} \to A_n \xrightarrow{d_n} A_{n-1} \to \cdots$ in $\mathscr{A}$ is *α-pure exact* if it is exact and $\operatorname{Ker} d_n \hookrightarrow A_n \twoheadrightarrow \operatorname{Im} d_n$ is α-pure for all $n \in \mathbb{Z}$.

An object Q in $\mathscr{A}$ is *α-pure projective* if $\operatorname{Hom}_R(Q, -)$ takes α-pure exact sequences to exact sequences, this is equivalent to say that Q is a retract of a direct sum of α-presentables.

The notions of α-pure projective resolution, α-pure projective dimension $\mathrm{ppd}_\alpha(A)$ of an object A in $\mathscr{A}$, etc. are defined in the obvious way. The *α-pure global dimension* of $\mathscr{A}$ is denoted by

$$\mathrm{pgd}_\alpha(\mathscr{A}) = \sup\{\mathrm{ppd}_\alpha(A) \mid A \text{ in } \mathscr{A}\}.$$

If $\mathscr{A} = \mathrm{Mod}(R)$ is the category of modules over a ring R we abbreviate $\mathrm{pgd}_\alpha(R) = \mathrm{pgd}_\alpha(\mathrm{Mod}(R))$.

Given A and B in $\mathscr{A}$, the *α-pure extension groups*

$$\mathrm{PExt}^n_{\alpha,\mathscr{A}}(A,B)$$

are defined as the cohomology of an α-pure projective resolution of A with coefficients in B.

Remark 3.10. Any object A in $\mathscr{A}$ is an α-filtered colimit of α-presentable objects $A = \mathrm{colim}_{\lambda\in\Lambda} P_\lambda$, hence the construction in Remark 2.8 yields an α-pure projective resolution of A, in particular

$$\mathrm{PExt}^n_{\alpha,\mathscr{A}}(A,B) = \lim_{\lambda\in\Lambda}{}^n \mathrm{Hom}_{\mathscr{A}}(P_\lambda, B).$$

If A is α-flat we can take P_λ projective for all $\lambda \in \Lambda$ and the projective resolution of A in Remark 2.8 is also α-pure, so $\mathrm{PExt}^n_{\alpha,\mathscr{A}}(A,B) = \mathrm{Ext}^n_{\mathscr{A}}(A,B)$ in this case. This proves that

$$\mathrm{fgd}_\alpha(\mathscr{A}) \leq \mathrm{pgd}_\alpha(\mathscr{A}).$$

For an arbitrary object A, the spectral sequence for the composition of functors $\mathrm{Hom}_{\mathscr{A}}(A,B) = \mathrm{Hom}_{\mathscr{A}}(\mathrm{colim}_{\lambda\in\Lambda} P_\lambda, B) = \lim_{\lambda\in\Lambda} \mathrm{Hom}_{\mathscr{A}}(P_\lambda, B)$ is of the form

$$E_2^{p,q} = \lim_{\lambda\in\Lambda}{}^p \mathrm{Ext}^q_{\mathscr{A}}(P_\lambda, B) \Longrightarrow \mathrm{Ext}^{p+q}_{\mathscr{A}}(A,B).$$

The comparison homomorphism between α-pure and ordinary extensions groups is part of this spectral sequence,

$$\mathrm{PExt}^n_{\alpha,\mathscr{A}}(A,B) = E_2^{n,0} \twoheadrightarrow E_\infty^{n,0} \subset \mathrm{Ext}^n_{\mathscr{A}}(A,B).$$

Lemma 3.11. *Any short exact sequence $A \hookrightarrow B \twoheadrightarrow C$ where C is α-flat is α-pure.*

Proof. Since $C = \mathrm{colim}_{\lambda\in\Lambda} P_\lambda$ is an α-filtered colimit of α-presentable projective objects, taking pullback along the canonical morphisms $P_\lambda \to C$

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, equal] & Q_\lambda \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\text{pull}"] & P_\lambda \arrow[d] \\ A \arrow[r, hook] & B \arrow[r, two heads] & C \end{tikzcd}$$

we can express the short exact sequence below as an α-filtered colimit

$$\operatorname*{colim}_{\lambda\in\Lambda}(A \hookrightarrow Q_\lambda \twoheadrightarrow P_\lambda)$$

of short exact sequences which split since P_λ is projective. $\square$

The following lemma admits the same proof than [Mur11, Theorem 20]. There, it is assumed that R is right noetherian or $\mathrm{card}\, R < \alpha$ but actually only α-coherence is used.

Lemma 3.12. *Let R be an α-coherent ring for some $\alpha > \aleph_0$. A complex X in $D(R)$ is α-compact if and only if $H_n(X)$ is an α-generated R-module for all $n \in \mathbb{Z}$.*

Lemma 3.13. *Let R be an α-coherent ring, $\alpha > \aleph_0$.*

(1) *The functor* $H_0\colon \mathrm{Mod}_\alpha(D(R)^\alpha) \to \mathrm{Mod}(R)$ *defined as* $H_0(F) = F(R)$ *takes projective objects to* α*-pure projective* R*-modules and preserves* α*-filtered colimits and* α*-pure exact sequences.*

(2) *The functor* $\mathbf{y}\colon \mathrm{Mod}(R) \subset D(R) \xrightarrow{S_\alpha} \mathrm{Mod}_\alpha(D(R)^\alpha)$ *takes* α*-pure projective* R*-modules to projective objects and preserves* α*-filtered colimits and* α*-pure exact sequences.*

Proof. If X is in $D(R)^\alpha$, then $H_0S_\alpha(X) = S_\alpha(X)(R) = D(X)(R,X) = H_0(R)$, which is α-presentable by the Lemma 3.13, hence H_0 takes projective objects to α-pure projective R-modules. In $\mathrm{Mod}_\alpha(D(R)^\alpha)$, α-filtered colimits are computed pointwise hence H_0 preserves these colimits. Since H_0 preserves split short exact sequences and α-filtered colimits, we deduce that it also preserves α-pure short (exact) sequences. This finishes the proof of (1).

If M is an α-presentable R-module, then M is α-compact in $D(R)$ by Lemma 3.12, so $S_\alpha(M)$ is projective in $\mathrm{Mod}_\alpha(D(R)^\alpha)$. It follows that $\mathbf{y}$ takes α-pure projective R-modules to projective objects.

Let $M = \mathrm{colim}_{\lambda\in\Lambda} M_\lambda$ be an α-filtered colimit of R-modules. Denote $\mathscr{S} \subset \mathscr{T}$ the full subcategory of objects X such that the natural morphism

$$(\operatorname*{colim}_{\lambda\in\Lambda} S_\alpha(M_\lambda))(X) = \operatorname*{colim}_{\lambda\in\Lambda} \mathscr{T}(X, M_\lambda) \longrightarrow \mathscr{T}(X, \operatorname*{colim}_{\lambda\in\Lambda} M_\lambda) = (S_\alpha(M))(X)$$

is an isomorphism. The category $\mathscr{S}$ contains $\Sigma^n R$, $n \in \mathbb{Z}$. Indeed, for $n \neq 0$ this morphism is $0 \to 0$ and for $n = 0$ it is the identity $\mathrm{colim}_{\lambda\in\Lambda} M_\lambda \to \mathrm{colim}_{\lambda\in\Lambda} M_\lambda$. The category of abelian groups is locally finitely presentable, hence α-filtered colimits commute with products of less than α objects. This shows that $\mathscr{S}$ is closed under coproducts of less than α objects. The category $\mathscr{S}$ is also closed under exact triangles by the five lemma. Therefore $\mathscr{S} = D(R)^\alpha$ and hence $\mathbf{y}$ preserves α-filtered colimits.

Any α-pure short exact sequence of R-modules is an α-filtered colimit of split ones. Since $\mathbf{y}$ preserves split short exact sequences and α-filtered colimits we deduce that $\mathbf{y}$ preserves α-pure (short) exact sequences. This concludes the proof of (2). □

Corollary 3.14. *Given an* α*-coherent ring* R*,* $\alpha > \aleph_0$*, and an* R*-module* $M = \mathrm{colim}_\lambda P_\lambda$ *expressed as an* α*-filtered colimit of* α*-presentable* R*-modules* P_λ*,*

$$\mathrm{Ext}^n_{\alpha,\mathscr{C}}(S_\alpha(M), F) = \lim_{\lambda\in\Lambda}{}^n F(P_\lambda).$$

In particular, for $F = S_\alpha(\Sigma^j N)$*,* $j \in \mathbb{Z}$*,*

$$\mathrm{Ext}^n_{\alpha,\mathscr{C}}(S_\alpha(M), S_\alpha(\Sigma^j N)) = \lim_{\lambda\in\Lambda}{}^n \mathrm{Ext}^j_R(P_\lambda, N).$$

Proof. Take the α-pure projective resolution of M in Remark 3.10. Applying S_α we obtain a projective resolution of $S_\alpha(M)$ by Lemma 3.13 (2). Using this resolution to compute $\mathrm{Ext}^n_{\alpha,\mathscr{C}}(S_\alpha(M), F)$ we obtain the equation in the statement. □

Proposition 3.15. *Given an* α*-coherent ring* R*,* $\alpha > \aleph_0$*, if* H *is a cohomological functor in* $\mathrm{Mod}_\alpha(D(R)^\alpha)$*, then* $\mathrm{ppd}_\alpha(H(R)) \leq \mathrm{pd}(H)$*. Moreover, for any* R*-module* M*,* $\mathrm{ppd}_\alpha(M) = \mathrm{pd}(S_\alpha(M))$*.*

Proof. By Remark 2.9 and Lemma 3.11, any projective resolution of H is also an α-pure projective resolution, hence Lemma 3.13 (1) proves the first part. Since $S_\alpha(M)(R) = M$, this also proves $\mathrm{ppd}_\alpha(M) \leq \mathrm{pd}(S_\alpha(M))$. The other inequality follows from Lemma 3.13 (2). □

Corollary 3.16. *If R is an α-coherent ring, then $\alpha > \aleph_0$, then*

$$\mathrm{pgd}_\alpha(R) \leq \mathrm{fgd}_\alpha(\mathrm{Mod}_\alpha(D(R)^\alpha)).$$

Moreover, if R is hereditary and $\mathrm{fgd}_\alpha(\mathrm{Mod}_\alpha(D(R)^\alpha)) \leq 2$, then the equality holds $\mathrm{fgd}_\alpha(\mathrm{Mod}_\alpha(D(R)^\alpha)) = \mathrm{pgd}_\alpha(R)$.

Proof. The first part follows directly from Proposition 3.15. By Corollary 2.13, if $\mathrm{fgd}_\alpha(\mathrm{Mod}_\alpha(D(R)^\alpha)) \leq 2$, then every α-flat object is representable and the result follows from Proposition 3.15 and the fact that, if R is hereditary, then any complex X splits as $X \cong \bigoplus_{n\in\mathbb{Z}} \Sigma^n H_n(X)$. □

We can now prove Proposition 3.7.

Proof of Proposition 3.7. Let S be the set of K-projective complexes formed by free R-modules of the form $\bigoplus_{i\in I} R$ with $\mathrm{card}\, I < \alpha$. By [Mur11, Theorem 15], any α-compact complex in $D(R)$ is isomorphic to an object in S. The morphism set between two of those free R-modules

$$\mathrm{Hom}_R(\bigoplus_{i\in I} R, \bigoplus_{j\in J} R) \cong \prod_{i\in I} \mathrm{Hom}_R(R, \bigoplus_{j\in J} R) \cong \prod_{i\in I}\bigoplus_{j\in J} \mathrm{Hom}_R(R,R) \cong \prod_{i\in I}\bigoplus_{j\in J} R$$

has cardinal $\leq \alpha^{\mathrm{card}\, I}$, and, under our assumptions, $\alpha^{\mathrm{card}\, I} \leq \alpha$, compare [Jec03, Theorem 5.20]. This shows that $\mathrm{card}\, S \leq \alpha$, and moreover that the set of chain maps between two objects X and Y in S has cardinal $\leq \alpha$. Since $D(R)(X,Y)$ is the quotient of the set of chain maps by the homotopy relation, we deduce that $\mathrm{card}\, D(R)^\alpha \leq \alpha$.

For the last part of the statement we use Proposition 2.12 and Corollaries 2.14 and 3.16. □

The following result gives a necessary condition for the representability of cohomological functors in $\mathrm{Mod}_\alpha(D(R)^\alpha)$ which fit into an extension of restricted representables.

Lemma 3.17. *Let M and N be R-modules and $S_\alpha(\Sigma^j N) \overset{a}{\hookrightarrow} F \overset{b}{\twoheadrightarrow} S_\alpha(M)$ an extension in $\mathrm{Mod}_\alpha(D(R)^\alpha)$, $j > 0$, classified by an element*

$$e_F \in \mathrm{Ext}^1_{\alpha,D(R)^\alpha}(S_\alpha(M), S_\alpha(\Sigma^j N)) = \lim_\lambda{}^1 \mathrm{Ext}^j_R(P_\lambda, N) = E_2^{1,j}.$$

Here $M = \mathrm{colim}_{\lambda\in\Lambda} P_\lambda$ is an α-filtered colimit of α-presentable R-modules. If $F = S_\alpha(X)$ for some X in $D(R)$, then the second differential of the spectral sequence in Remark 3.10 maps e_F to zero,

$$d_2\colon E_2^{1,j} \longrightarrow E_2^{3,j-1}, \quad d_2(e_F) = 0.$$

Proof. The spectral sequence in Remark 3.10 identifies with the Adams spectral sequence in Section 6.8 below abutting to $D(R)(M, \Sigma^j N) = \mathrm{Ext}^j_R(M,N)$ via the second equation in Corollary 3.14. Hence, the statement follows from Theorem 7.1 and the fact that the following morphism is injective for $p = 3$ and $q = -1$,

$$\begin{aligned} \mathrm{Ext}^{p,q}_{\alpha,D(R)^\alpha}(S_\alpha(M), S_\alpha(\Sigma^j N)) &\longrightarrow \mathrm{Ext}^{p,q}_{\alpha,D(R)^\alpha}(F,F), \\ x &\mapsto a\cdot x\cdot b. \end{aligned} \tag{3.18}$$

We show that it is injective for $p \geq 0$ and $q < 0$. Indeed, this morphism decomposes as

$$\mathrm{Ext}^{p,q}_{\alpha,D(R)^\alpha}(S_\alpha(M), S_\alpha(\Sigma^j N)) \xrightarrow{a\cdot -} \mathrm{Ext}^{p,q}_{\alpha,D(R)^\alpha}(S_\alpha(M), F) \xrightarrow{-\cdot b} \mathrm{Ext}^{p,q}_{\alpha,D(R)^\alpha}(F,F).$$

The kernel of the first arrow is the image of a morphism from

$$\operatorname{Ext}^{p-1,q}_{\alpha,D(R)^\alpha}(S_\alpha(M),S_\alpha(M)) = \lim_{\lambda}{}^{p-1}\operatorname{Ext}^q_R(M_\lambda,M)=0,$$

which vanishes since $q<0$. The kernel of the second arrow is the image of a morphism from the middle term of the following exact sequence

$$\begin{gathered}\operatorname{Ext}^{p-1,q}_{\alpha,D(R)^\alpha}(S_\alpha(\Sigma^j N),S_\alpha(\Sigma^j N))\\ \downarrow\\ \operatorname{Ext}^{p-1,q}_{\alpha,D(R)^\alpha}(S_\alpha(\Sigma^j N),F)\\ \downarrow\\ \operatorname{Ext}^{p-1,q}_{\alpha,D(R)^\alpha}(S_\alpha(\Sigma^j N),S_\alpha(M))\end{gathered}$$

which vanishes since

$$\begin{aligned}\operatorname{Ext}^{p-1,q}_{\alpha,D(R)^\alpha}(S_\alpha(\Sigma^j N),S_\alpha(\Sigma^j N)) &= \operatorname{Ext}^{p-1,q}_{\alpha,D(R)^\alpha}(S_\alpha(N),S_\alpha(N))\\ &= \lim_{\lambda}{}^{p-1}\operatorname{Ext}^q_R(N_\lambda,N)=0,\\ \operatorname{Ext}^{p-1,q}_{\alpha,D(R)^\alpha}(S_\alpha(\Sigma^j N),S_\alpha(M)) &= \operatorname{Ext}^{p-1,q}_{\alpha,D(R)^\alpha}(S_\alpha(N),S_\alpha(\Sigma^{-j} M))\\ &= \lim_{\lambda}{}^{p-1}\operatorname{Ext}^{q-j}_R(N_\lambda,M)=0.\end{aligned}$$

Here we use that $q<0<J$. □

As a consequence, we obtain a sufficient condition for the existence of non-representable cohomological functors in $\operatorname{Mod}_\alpha(D(R)^\alpha)$.

Proposition 3.19. *Let R be an α-coherent ring. If there is an R-module N with injective dimension ≤ 1 but $\operatorname{PExt}^n_{\alpha,R}(M,N)\neq 0$ for some R-module M and some $n\geq 3$, then ARO_α fails for $D(R)$.*

Proof. If $n>3$ we can take an α-pure short exact sequence $M'\hookrightarrow P\twoheadrightarrow M$ with α-pure projective P, so $\operatorname{PExt}^n_{\alpha,R}(M,N)\cong\operatorname{PExt}^{n-1}_{\alpha,R}(M',N)$, hence we may assume that $n=3$.

By Lema 3.17 it is enough to show that $d_2\colon E_2^{1,1}\to E_2^{3,0}$ is non-trivial. The target is non-trivial $E_2^{3,0}=\operatorname{PExt}^3_{\alpha,R}(M,N)\neq 0$. By degree reasons, there are no non-trivial differentials out of $E_n^{3,0}$, hence $E_2^{3,0}$ surjects onto $E_\infty^{3,0}\subset\operatorname{Ext}^3_R(M,N)=0$. Therefore, all elements in $E_2^{3,0}$ must be in the image of an incoming differential. Since $E_2^{0,2}=\lim_\lambda\operatorname{Ext}^2(P_\lambda,N)=0$, then $E_3^{0,2}=0$ and the only possibly non-trivial incoming differential is $d_2\colon E_2^{1,1}\to E_2^{3,0}$, which must be surjective. □

We finally prove Theorem 3.3.

Proof of Theorem 3.3. The first part of the statement follows from Corollary 1.3 and Proposition 3.15. If R is hereditary, any complex X splits as a direct sum of its shifted homologies $X\cong\bigoplus_{n\in\mathbb{Z}}\Sigma^n H_n(X)$. Therefore, on the one hand, (2) follows from Corollary 2.5 and Proposition 3.15, and on the other hand (1) is an immediate consequence of Corollaries 2.13 and 3.16 and Proposition 3.19. □

4. On $\aleph_1$-Adams representability for objects and the continuum hypothesis

We already know by Corollary 2.14 that if $\mathscr{T}$ is an $\aleph_1$-compactly generated triangulated category with $\operatorname{card}\mathscr{T}^{\aleph_1} = \aleph_1$, then $\mathscr{T}$ satisfies $\mathrm{ARO}_{\aleph_1}$. We have applied this result to derived categories of rings (Proposition 3.7). In this section, we give further examples *assuming the continuum hypothesis*.

4.1. **Stable homotopy category of spectra.** The stable homotopy category of spectra $\mathscr{T} = \mathrm{Ho}(\mathrm{Sp})$ is $\aleph_0$-compactly generated and $\mathrm{Ho}(\mathrm{Sp})^{\aleph_0} \leq \aleph_0 < \aleph_1$. Then $\mathrm{Ho}(\mathrm{Sp})^{\aleph_1} = \aleph_1$ by Corollary 8.5.

4.2. **Homotopy category of projectives modules.** Let $\mathscr{T} = K(\text{Proj-}R)$ be the homotopy category of complexes of projective (right) modules over a ring R of $\operatorname{card} R \leq \aleph_1$. This category is often not $\aleph_0$-compactly generated, but it is always $\aleph_1$-compactly generated, cf. [Nee08].

Proposition 4.1. *Under the continuum hypothesis* $\operatorname{card} K(\text{Proj-}R)^{\aleph_1} \leq \aleph_1$.

Proof. By [Nee08, Theorem 5.9], a complex of projective R-modules is $\aleph_1$-compact in $K(\text{Proj-}R)$ if and only if it is isomorphic in $K(R\text{-Proj})$ to a complex of free R-modules with $< \aleph_1$ generators. Since we are assuming the continuum hypothesis and $\operatorname{card} R \leq \aleph_1$, we can proceed exactly as in the proof of Proposition 3.7. $\square$

4.3. **Homotopy category of injectives modules.** Let R be a right noetherian ring of $\operatorname{card} R \leq \aleph_1$. The homotopy category $\mathscr{T} = K(\text{Inj-}R)$ of injective (right) R-modules is $\aleph_0$-compactly generated [Kra05].

Proposition 4.2. *Under the continuum hypothesis* $\operatorname{card} K(\text{Inj-}R)^{\aleph_0} \leq \aleph_1$.

Proof. By [Kra05], $K(\text{Inj-}R)^{\aleph_0}$ is equivalent to the derived category $D^b(\operatorname{mod}(R))$ of bounded complexes of finitely presentable R-modules. Since R is right noetherian, $D^b(\operatorname{mod}(R))$ is equivalent to the full subcategory of $K(\text{Proj-}R)^{\aleph_0}$ spanned by bounded below complexes of finitely presentable projective R-modules with bounded cohomology. Now proceed as in the proof of Proposition 3.7. $\square$

4.4. **Derived category of sheaves on a non-compact manifold.** Let M be a connected paracompact manifold and $D(\mathrm{Sh}/M)$ the derived category of the abelian category Sh/M of sheaves of abelian groups over M. Neeman [Nee01a] proved that if M is non-compact, connected and $\dim M \geq 1$, then $D(\mathrm{Sh}/M)$ has no non-zero compact object, so it cannot be $\aleph_0$-compactly generated.

Proposition 4.3. *The category $D(\mathrm{Sh}/M)$ is $\aleph_1$-compactly generated and, under the continuum hypothesis,* $\operatorname{card} D(\mathrm{Sh}/M)^{\aleph_1} \leq \aleph_1$.

Proof. Since M is paracompact, we can take a countable basis $\{U_i\}_{i\in I}$ of open sets of M such that $U_i \cap U_j$ is contractible for all $i, j \in I$, e.g. put a Riemannian metric on M and take a countable basis of geodesically convex balls. By [Gro57, Section 1.9], a set of generators of Sh/M is given by $\{\mathbb{Z}_{U_i}\}_{i\in I}$, where $\mathbb{Z}_{U_i}$ is the extension by zero of the constant sheaf $\mathbb{Z}$ on U_i. Let $\mathscr{R}$ be the full subcategory of Sh/M spanned by these sheaves. It has countably many objects. Moreover, since each U_j is connected, the monomorphisms $\mathbb{Z}_{U_i} \to \mathbb{Z}_M$ show that $\operatorname{Hom}(\mathbb{Z}_{U_j}, \mathbb{Z}_{U_i}) = \mathbb{Z}_{U_i}(U_j) \subset \mathbb{Z}_M(U_j) = \{\text{locally constant maps } U_j \to \mathbb{Z}\} = \mathbb{Z}$ is countable for all $i, j \in I$.

The countable category $\mathscr{R}$ can be regarded as a ring with several objects. The derived category $D(\mathrm{Sh}/M)$ is a Bousfield localization $D(\mathrm{Sh}/M) = D(\mathscr{R})/\mathscr{L}_{\mathrm{Sh}/M}$ [AJS00, Proposition 5.1]. Since $\operatorname{card}\mathscr{R} < \aleph_1$, the many object version of [Mur11, Theorem 20] proves that the generators of the localizing subcategory $\mathcal{L}_{\mathrm{Sh}/M}$ described in the proof of [AJS00, Proposition 5.1] are $\aleph_1$-compact. Hence $D(\mathrm{Sh}/M)$ is $\aleph_1$-compactly generated by [Nee01b, Theorem 4.4.9], and the subcategory of $\aleph_1$-compact objects is $D(\mathrm{Sh}/M)^{\aleph_1} = D(\mathscr{R})^{\aleph_1}/\mathscr{L}^{\aleph_1}_{\mathrm{Sh}/M}$.

Now, let us assume the continuum hypothesis. The many objects version of Proposition 3.7 shows that $\operatorname{card} D(\mathscr{R})^{\aleph_1} \leq \aleph_1$, and the explicit description of the Verdier quotient $D(\mathrm{Sh}/M)^{\aleph_1} = D(\mathscr{R})^{\aleph_1}/\mathscr{L}^{\aleph_1}_{\mathrm{Sh}/M}$ proves that $\operatorname{card} D(\mathrm{Sh}/M)^{\aleph_1} \leq \aleph_1$ too. $\square$

4.5. **Stable motivic homotopy category.** Let S be a noetherian scheme of finite Krull dimension. The stable motivic homotopy category $\mathrm{SH}(S)$ of Morel and Voevodsky is a compactly generated triangulated category which intuitively models a homotopy theory of schemes over S where the affine line $\mathbb{A}^1$ plays the role of the unit interval in classical homotopy theory. In practice, we start with the category Sm/S of smooth schemes of finite type over S endowed with the Nisnevich topology. We perform two left Bousfield localizations on the category of simplicial presheaves on Sm/S, one to turn homotopy sheaves into weak equivalences and another one to contract the affine line $\mathbb{A}^1$. Then we consider spectra with respect to the suspension functor defined by smashing with the projective line $\mathbb{P}^1 \simeq \mathbb{S}^1 \wedge (\mathbb{A}^1 - 0)$ pointed at ∞. This yields a stable model category whose homotopy category is $\mathrm{SH}(S)$.

It was stated in [Voe98, Proposition 5.5] and proved in [NS11, Theorem 13] that if S can be covered by spectra of countable rings, then $\operatorname{card}\mathrm{SH}(S)^{\aleph_0} \leq \aleph_0 < \aleph_1$, hence under the continuum hypothesis $\operatorname{card}\mathrm{SH}(S)^{\aleph_1} \leq \aleph_1$, see Corollary 8.5. The results in [NS11] extend straightforwardly to show that, if S can be converted by spectra of rings of cardinal $\leq \aleph_1$, then $\operatorname{card}\mathrm{SH}(S)^{\aleph_0} \leq \aleph_1$. Therefore $\operatorname{card}\mathrm{SH}(S)^{\aleph_1} \leq \aleph_1$ under the continuum hypothesis, again by Corollary 8.5.

5. Neeman's conjecture on Rosický functors

The following definition is due to Neeman [Nee09, Definition 1.19].

Definition 5.1. Let $\mathscr{T}$ be a triangulated category with (co)products. A *Rosický functor* is a functor $H\colon \mathscr{T} \to \mathscr{A}$ to an abelian category with (co)products which takes exact triangles to exact sequences, is full, reflects isomorphisms, preserves (co)products, and there is a small full subcategory $\mathscr{P} \subset \mathscr{T}$ closed under (de)suspensions, formed by α-small objects in $\mathscr{T}$, and such that $\{H(P) \mid P \in \operatorname{Ob}\mathscr{P}\}$ is a set of projective generators of $\mathscr{A}$ and H induces a bijection $\mathscr{T}(P,X) \cong \mathscr{A}(H(P),H(X))$ whenever P is in $\mathscr{P}$.

Under the standing assumptions of Section 1, the restricted Yoneda functor S_α satisfies all properties of a Rosický functor except for being full, the subcategory $\mathscr{C}$ consists of representable functors. Moreover, if $\mathscr{C} = \mathscr{T}^\alpha$, S_α is a Rosický functor if and only if ARM_α holds.

Neeman conjectured that a triangulated category has a Rosický functor if and only if it is well generated. It is easy to see that, if $\mathscr{T}$ has a Rosický functor, then it is well generated, we give a proof, first discovered by Rosický, in this section. Neeman's conjecture is still open in the other direction. A consequence of Corollary

5.3 is that it is enough to look for Rosický functors of the form S_α for an appropriate $\mathscr{C}$. Example 3.4 shows that we cannot always take $\mathscr{C} = \mathscr{T}^\alpha$ for some α, which was the experts' first guess. Nevertheless, it is still an open question whether categories such as $D(k[[x, y]])$ possess a Rosický functor.

Proposition 5.2. *Let $\mathscr{T}$ be a triangulated category with coproducts. If there exists $H\colon \mathscr{T} \to \mathscr{A}$ be a Rosický functor, then the category $\mathscr{T}$ is well generated. Moreover, if $\mathscr{C}$ is the completion of $\mathscr{P}$ by coproducts of $< \alpha$ objects, then $\mathscr{C}$ satisfies assumptions (1–3) in Section 1 and S_α factors as*

$$S_\alpha\colon \mathscr{T} \xrightarrow{H} \mathscr{A} \xrightarrow{i} \mathrm{Mod}_\alpha(\mathscr{C}),$$

where i is fully faithful and exact.

Proof. Let us first show that $\mathscr{T}$ is well generated. This fact was first discovered by Rosický (unpublised). We follow Krause's criterion saying that a triangulated category is well generated if and only if it satisfies conditions (G1–G3), cf. [Kra01]. The set of objects of $\mathscr{P}$ clearly satisfies (G1) and (G3). We now check (G2). Let $\{f_i\colon X_i \to Y_i\}_{i\in I}$ be a set of morphisms in $\mathscr{T}$ such that $\mathscr{T}(P, f_i)$ is an epimorphism for all $i \in I$ and P in $\mathscr{P}$. Since $\mathscr{T}(P, f_i) = \mathscr{A}(H(P), H(f_i))$ and the objects $H(P)$ form a set of projective generators, $H(f_i)$ is an epimorphism in $\mathscr{A}$ for all $i \in I$. In an abelian category, a coproduct of epimorphisms is an epimorphism. Since H preserves coproducts we deduce that $H(\coprod_{i\in I} f_i)$ is an epimorphism, hence, $\mathscr{T}(P, \coprod_{i\in I} f_i) = \mathscr{A}(H(P), H(\coprod_{i\in I} f_i))$ is surjective for all P in $\mathscr{P}$. This proves (G2). By Krause's criterion we also know that $\mathscr{P} \subset \mathscr{T}^\alpha$, therefore $\mathscr{C}$ satisfies (1–3) in Section 1.

The functor i is defined by $i(A) = \mathscr{A}(H(-), A)$. This $\mathscr{C}$-module is α-continuous since H preserves coproducts. The properties of Rosický functors show that H induces an equivalence between $\mathscr{C}$ and its full image in $\mathscr{A}$. Hence $\{H(C) \mid C \in \mathrm{Ob}\,\mathscr{C}\}$ is also a set of projective generators of $\mathscr{A}$ and i is fully faithful. The composite iH is naturally isomorphic to S_α since for any X in $\mathscr{T}$ and any coproduct $\coprod_{i\in I} P_i$ with P_i in $\mathscr{P}$ and $\mathrm{card}\, I < \alpha$,

$$\begin{aligned} S_\alpha(X)(\coprod_{i\in I} P_i) &= \mathscr{T}(\coprod_{i\in I} P_i, X) = \prod_{i\in I} \mathscr{T}(P_i, X) \overset{H}{\cong} \prod_{i\in I} \mathscr{A}(H(P_i), H(X)) \\ &= \mathscr{A}(\bigoplus_{i\in I} H(P_i), H(X)) = \mathscr{A}(H(\coprod_{i\in I} P_i), H(X)) = iH(X)(\coprod_{i\in I} P_i). \end{aligned}$$

□

Corollary 5.3. *A triangulated category $\mathscr{T}$ admits a Rosický functor if and only if it is well generated and S_α is full for some $\mathscr{C} \subset \mathscr{T}$ satisfying (1–3) in Section 1.*

Recall that Corollary 2.4 gives us a criterion for the restricted Yoneda functor S_α to be full.

Remark 5.4. Let Q be a quiver without oriented cycles, k an uncountable field, kQ its path algebra over k, which is hereditary, and α any regular cardinal. As we showed in Example 3.4, for $\mathscr{T} = D(kQ)$ and $\mathscr{C} = \mathscr{T}^\alpha$ the functor S_α is never a Rosický functor. Nevertheless, if R is any hereditary ring, the homology functor $H_*\colon D(R) \to \mathrm{Mod}(R)^{\mathbb{Z}}$ to the category of $\mathbb{Z}$-graded R-modules is a Rosický functor for $\mathscr{P}$ the full subcategory spanned by $\{\Sigma^n R\}_{n\in\mathbb{Z}}$, here $\alpha = \aleph_0$.

These are only known Rosický functors different from the restricted Yoneda functor S_α with $\mathscr{C} = \mathscr{T}^\alpha$ for $\mathscr{T}$ a category satisfying ARM_α. Triangulated categories possessing a Rosický functor satisfy further properties of interest, e.g. the Brown representability theorem for the dual, see [Nee09]. Hence it would be interesting to know if there are more kinds of Rosický functors.

6. Obstruction theory in triangulated categories

Recall that we are under the standing assumptions of Section 1. In diagrams, the degree of a homogeneous morphism in $\mathscr{T}$ or $\mathrm{Mod}_\alpha(\mathscr{C})$ is indicated by a label in the arrow, e.g.

$$X \xrightarrow[+n]{f} Y$$

is a morphism $f\colon X \to \Sigma^n Y$. We mostly consider homogeneous morphisms. We do not explicitly indicate the degree when it is 0, when it is understood, or when it is irrelevant. Hence an exact triangle $X \to Y \to Z \to \Sigma X$ in $\mathscr{T}$ looks like

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \arrow[dl, "i"] \\ & Z. \arrow[ul, "q", "+1"'] & \end{tikzcd}$$

6.1. **Phantom maps and cellular objects.**

Definition 6.1. A morphism $f\colon X \to Y$ in $\mathscr{T}$ is a *phantom map* if $S_\alpha(f) = 0$. Moreover, f is an *n-phantom map* if it decomposes as a product of n ordinary phantom maps, i.e. $f = f_1 \cdots f_n$ with f_i phantom, $1 \le i \le n$. An *∞-phantom map* is a morphism f which is an n-phantom map for all $n > 0$.

The following result is a consequence of the fact that S_α takes exact triangles to exact sequences.

Lemma 6.2. *In an exact triangle*

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \arrow[dl, "i"] \\ & Z \arrow[ul, "q"] & \end{tikzcd}$$

where we deliberately do not specify which morphism is of degree $+1$*, the following statements are equivalent:*

- f *is a phantom map.*
- $S_\alpha(i)$ *is a monomorphism.*
- $S_\alpha(q)$ *is an epimorphism.*
- $S_\alpha(Y) \overset{S_\alpha(i)}{\hookrightarrow} S_\alpha(Z) \overset{S_\alpha(q)}{\twoheadrightarrow} S_\alpha(X)$ *is a short exact sequence.*

Remark 6.3. Phantom maps form an ideal $\mathscr{I} \subset \mathscr{T}$ and n-phantom maps form its n^{th} power ideal, $\mathscr{I}^n = \mathscr{I} \overset{n}{\cdots} \mathscr{I} \subset \mathscr{T}$. Moreover, ∞-phantom maps are the intersection ideal

$$\mathscr{I}^\infty = \bigcap_{n>0} \mathscr{I}^n \subset \mathscr{T}.$$

Definition 6.4. A 0-*cellular object* is a trivial object in $\mathscr{T}$. Moreover, X is n-*cellular* for $n > 0$ if it is a retract of an object X' fitting into an exact triangle

$$\begin{tikzcd} P \arrow[rr] & & Y \arrow[dl] \\ & X' \arrow[ul, "+1"] & \end{tikzcd}$$

where Y is $(n-1)$-cellular and P is in $\operatorname{Add}(\mathscr{C})$.

Proposition 6.5. *Let $1 \leq n \leq \infty$. A morphism $f\colon X \to Y$ in $\mathscr{T}$ is an n-phantom map if and only if for any morphism $g\colon Z \to X$ from an n-cellular object Z we have $fg = 0$. Moreover, Z is an n-cellular object if and only if for any morphism $g\colon Z \to X$ and any n-phantom map $f\colon X \to Y$ we have $fg = 0$.*

Since $\mathscr{C}$ is essentially small, $(\operatorname{Add}(\mathscr{C}), \mathscr{I})$ is a projective class by [Chr98, Lemma 3.2], hence Proposition 6.5 follows from [Chr98, Theorem 3.5].

6.2. **Adams and Postnikov resolutions.** Adams resolutions go back to Adams' construction of the spectral sequence that bears his name. The definition below is due to Christensen, cf. [Chr98].

Definition 6.6. An *Adams resolution* (X, W_*, P_*) of an object X in $\mathscr{T}$ is a countable sequence of exact triangles

$$\begin{tikzcd}
X \arrow[rr, "j_0"] & & W_0 \arrow[rr, "j_1"] \arrow[dl, "r_0", "+1"'] & & W_1 \arrow[rr, "j_2"] \arrow[dl, "r_1", "+1"'] & & W_2 \arrow[rr, "j_3"] \arrow[dl, "r_2", "+1"'] & & W_3 \arrow[dl, "r_3", "+1"'] & \cdots \\
& P_0 \arrow[ul, "g_0"] & & P_1 \arrow[ul, "g_1"] & & P_2 \arrow[ul, "g_2"] & & P_3 \arrow[ul, "g_3"] & &
\end{tikzcd}$$

such j_n is a phantom map and P_n is in $\operatorname{Add}(\mathscr{C})$, $n \geq 0$.

Remark 6.7. An Adams resolution of X can be easily constructed by induction. We start with an epimorphism from a projective object $S_\alpha(P_0) \twoheadrightarrow S_\alpha(X)$, i.e. P_0 is in $\operatorname{Add}(\mathscr{C})$. This morphism is represented by a unique $g_0\colon P_0 \to X$. If we extend g_0 to an exact triangle we obtain r_0 and j_0, which is a phantom map by Lemma 6.2. If we have constructed the first n triangles we take an epimorphism from a projective object $S_\alpha(P_n) \twoheadrightarrow S_\alpha(W_{n-1})$ and proceed in the same way.

By Lemma 6.2, for any Adams resolution (X, W_*, P_*) the restricted Yoneda functor S_α maps

$$0 \longleftarrow X \xleftarrow[g_0]{} P_0 \xleftarrow[r_0 g_1]{+1} P_1 \xleftarrow[r_1 g_2]{+1} P_2 \xleftarrow[r_2 g_3]{+1} P_3 \longleftarrow \cdots$$

to a projective resolution of $S_\alpha(X)$ in $\operatorname{Mod}_\alpha(\mathscr{C})$.

Postnikov resolutions are an enrichment of Benson–Krause–Schwede's Postnikov systems that we recall in Definition 6.28 below, cf. [BKS04].

Definition 6.8. A *Postnikov resolution* (X, X_*, P_*) of an object X in $\mathscr{T}$ is a diagram

$$\begin{tikzcd}
X & & & & & & & & & \cdots \\
0 \arrow[u] \arrow[rr, "i_0"] & & X_0 \arrow[ull, bend right, "p_0"', "+1"] \arrow[rr, "i_1"] \arrow[dl, "q_0", "+1"'] & & X_1 \arrow[ullll, bend right, "p_1"', "+1"] \arrow[rr, "i_2"] \arrow[dl, "q_1", "+1"'] & & X_2 \arrow[ullllll, bend right, "p_2"', "+1"] \arrow[rr, "i_3"] \arrow[dl, "q_2", "+1"'] & & X_3 \arrow[ullllllll, bend right, "p_3"', "+1"] \arrow[dl, "q_3", "+1"'] & \cdots \\
& P_0 \arrow[ul, "f_0"] & & P_1 \arrow[ul, "f_1"] & & P_2 \arrow[ul, "f_2"] & & P_3 \arrow[ul, "f_3"] & &
\end{tikzcd}$$

consisting of a countable sequence of exact triangles and commutative triangles, $p_n = p_{n+1} i_{n+1}$, $n \geq 0$, such that S_α maps

$$\text{(6.9)} \qquad 0 \longleftarrow X \xleftarrow[p_0 q_0^{-1}]{} P_0 \xleftarrow[q_0 f_1]{+1} P_1 \xleftarrow[q_1 f_2]{+1} P_2 \xleftarrow[q_2 f_3]{+1} P_3 \longleftarrow \cdots$$

to a projective resolution of $S_\alpha(X)$. In particular X_n is $(n+1)$-cellular.

We will denote the structure morphisms by f_n^X, i_n^X, q_n^X, and p_n^X when we need to distinguish between different Postnikov resolutions.

Lemma 6.10. *Given an object X in $\mathscr{T}$ and an Adams resolution (X, W_*, P_*), there exists a Postnikov resolution (X, X_*, P_*) fitting in octahedra as follows, $n \geq 0$,*

$$\begin{tikzcd}
W_{n-1} \arrow[rr, "j_n"] \arrow[ddd, "\phi_{n-1}"'] & & W_n \arrow[ddl, "r_n", "+1"'] \arrow[ddd, "\phi_n"] \\
& X \arrow[ul, "j_{n-1}\cdots j_0"'] \arrow[ur, "j_n\cdots j_0"] & \\
& P_n \arrow[uul, "g_n"] \arrow[dl, "f_n"] & \\
X_{n-1} \arrow[uur, "p_{n-1}", "+1"'] \arrow[rr, "i_n"'] & & X_n \arrow[uul, "p_n"', "+1"] \arrow[ul, "q_n", "+1"']
\end{tikzcd}$$

Here, for $n = 0$ we use the convention $X_{-1} = 0$, $W_{-1} = X$, and $X \to W_{-1}$ the identity morphism. Conversely, if a Postnikov resolution (X, X_*, P_*) is given, then there exists an Adams resolution (X, W_*, P_*) fitting in octahedra as above.

Proof. The Postnikov resolution together with the octahedra are constructed inductively. The step $n = 0$ is essentially given in the statement. We just need to choose a degree $+1$ isomorphism q_0, e.g. $X_0 = \Sigma P_0$ and q_0 the identity. In the n^{th} step, we first complete $f_n = \phi_{n-1} g_n$ to an exact triangle, this yields i_n and q_n. Then we obtain ϕ_n and p_n by applying the octahedral axiom.

Let us tackle the converse. The Adams resolution together with the octahedra are also defined by induction. For the step $n = 0$, we just need to complete $g_0 = p_0 q_0^{-1}$ to an exact triangle. This yields j_0, r_0 and $\phi_0 = q_0^{-1} r_0$. Notice that j_0 is a phantom map since $S_\alpha(p_0)$ is an epimorphism in $\mathrm{Mod}_\alpha(\mathscr{C})$.

In the n^{th} step, we first complete p_n to an exact triangle, this yields ϕ_n and the morphism $X \to W_n$, which a fortiori will be $j_n \cdots j_0$ (so far we do not have a j_n). We also obtain $r_n = q_n \phi_n$. We then apply the octahedral axiom. This produces g_n and j_n. We must check that j_n is a phantom, or equivalently that $S_\alpha(g_n)$ is an epimorphism.

For $n = 1$, we have an exact sequence

$$0 \longleftarrow S_\alpha(X) \xleftarrow{S_\alpha(p_0 q_0^{-1})} S_\alpha(P_0) \xleftarrow[+1]{S_\alpha(q_0 f_1)} S_\alpha(P_1).$$

Since q_0 is an isomorphism, $\operatorname{Im} S_\alpha(f_1) = \operatorname{Ker} S_\alpha(p_0)$ and the triangle

$$\begin{tikzcd}
X \arrow[dr, "j_0"'] & & X_0 \arrow[ll, "+1", "p_0"'] \\
& W_0 \arrow[ur, "\phi_0"'] &
\end{tikzcd}$$

implies that $\operatorname{Ker} S_\alpha(p_0) = \operatorname{Im} S_\alpha(\phi_0)$. Since j_0 is phantom $S_\alpha(\phi_0)$ is a monomorphism. Hence, $S_\alpha(g_1)$ must an epimorphism since $f_1 = \phi_0 g_1$ and $\operatorname{Im} S_\alpha(f_1) = \operatorname{Im} S_\alpha(\phi_0)$.

Let $n > 1$. By induction hypothesis, for $0 \leq k < n$, j_k is a phantom and the sequences

$$\begin{tikzcd} 0 & S_\alpha(W_{k-1}) \arrow[l] & S_\alpha(P_k) \arrow[l, two heads, "S_\alpha(g_k)"'] & S_\alpha(W_k) \arrow[l, hook', "S_\alpha(r_k)"', "+1"] & 0 \arrow[l] \end{tikzcd}$$

are short exact. Moreover, in the following diagram

$$\begin{tikzcd}[column sep=small] S_\alpha(P_{n-2}) & & S_\alpha(P_{n-1}) \arrow[ll, "S_\alpha(q_{n-2}f_{n-1})"', "+1"] \arrow[dl, two heads, "S_\alpha(g_n-1)"] & & S_\alpha(P_n) \arrow[ll, "S_\alpha(q_{n-1}f_n)"', "+1"] \arrow[dl, two heads, "S_\alpha(g_n)"] \\ & S_\alpha(W_{n-2}) \arrow[ul, hook', "+1"', "S_\alpha(r_{n-2})"] & & S_\alpha(W_{n-1}) \arrow[ul, hook', "+1"', "S_\alpha(r_{n-1})"] & \end{tikzcd}$$

the horizontal row is also an exact sequence. Hence, $S_\alpha(W_{n-1}) = \operatorname{Im} S_\alpha(q_{n-2}f_{n-1})$ and therefore $S_\alpha(g_n)$ must be an epimorphism. $\square$

Corollary 6.11. *Any object X in $\mathscr{T}$ has a Postnikov resolution.*

This follows from Lemma 6.10 and the fact that any object in $\mathscr{T}$ has an Adams resolution, see Remark 6.7.

6.3. **Postnikov resolutions and ∞-phantom maps.** In this section we define a homotopy category of Postnikov resolutions. This is one of the key ingredients of our obstruction theory.

Definition 6.12. A *morphism of Postnikov resolutions*

$$(h, \psi_*, \varphi_*)\colon (X, X_*, P_*) \longrightarrow (Y, Y_*, Q_*) \tag{6.13}$$

is given by morphisms $h\colon X \to Y$, $\psi_n\colon X_n \to Y_n$, $\varphi_n\colon P_n \to Q_n$ in $\mathscr{T}$, $n \geq 0$, such that the obvious triangles and squares commute,

$$\begin{tikzcd}[column sep=small] X \arrow[ddd, "h"'] & & & & & & & & \\ & 0 \arrow[ul] \arrow[rr] \arrow[ddd] & & X_0 \arrow[ulll, bend right] \arrow[rr] \arrow[dl] \arrow[ddd, "\psi_0"] & & X_1 \arrow[ulllll, bend right] \arrow[rr] \arrow[dl] \arrow[ddd, "\psi_1"] & & X_2 \arrow[ulllllll, bend right] \arrow[dl] \arrow[ddd, "\psi_2"] & \cdots \\ & & P_0 \arrow[ul] \arrow[ddd, "\varphi_0"] & & P_1 \arrow[ul] \arrow[ddd, "\varphi_1"] & & P_2 \arrow[ul] \arrow[ddd, "\varphi_2"] & & \\ Y & & & & & & & & \\ & 0 \arrow[ul] \arrow[rr] & & Y_0 \arrow[ulll, bend right] \arrow[rr] \arrow[dl] & & Y_1 \arrow[ulllll, bend right] \arrow[rr] \arrow[dl] & & Y_2 \arrow[ulllllll, bend right] \arrow[dl] & \cdots \\ & & Q_0 \arrow[ul] & & Q_1 \arrow[ul] & & Q_2 \arrow[ul] & & \end{tikzcd}$$

A pair of morphisms of Postnikov resolutions

$$(h, \psi_*, \varphi_*), (\bar{h}, \bar{\psi}_*, \bar{\varphi}_*)\colon (X, X_*, P_*) \longrightarrow (Y, Y_*, Q_*)$$

are *homotopic* $(h, \psi_*, \varphi_*) \simeq (\bar{h}, \bar{\psi}_*, \bar{\varphi}_*)$ if, for all $n > 0$, the following equivalent conditions hold:

(1) $\psi_n i_n^X = \bar{\psi}_n i_n^X$,
(2) $i_n^Y \psi_{n-1} = i_n^Y \bar{\psi}_{n-1}$,

(3) $\psi_n - \bar{\psi}_n$ factors through $q_n^X : X_n \to P_n$,
(4) $\psi_{n-1} - \bar{\psi}_{n-1}$ factors through $f_n^Y : Q_n \to Y_{n-1}$.

This natural equivalence relation is additive: two morphisms are homotopic iff their difference $(h - \bar{h}, \psi_* - \bar{\psi}_*, \varphi_* - \bar{\varphi}_*)$ is nullhomotopic, i.e. homotopic to the trivial map. We denote $\mathbf{Pres}_\infty$ the category of Postnikov resolutions and $\mathbf{Pres}_\infty^{\simeq}$ its homotopy category. Both of them are additive.

The following theorem is the main result of this section. It establishes the existence of a functor with a certain property. Usually, when defining a functor, the complicated part is to show that composition is preserved. In this case the complicated part is the definition of the functor on morphisms, once this is achieved compatibility with composition is obvious.

Theorem 6.14. *There exists an essentially unique functor*

$$\Psi \colon \mathscr{T} \longrightarrow \mathbf{Pres}_\infty^{\simeq}$$

sending an object X to a Postnikov resolution $\Psi(X)$ of X and a map $h \colon X \to Y$ to the homotopy class $\Psi(h)$ of a morphism with first coordinate h. This functor is additive, full and essentially surjective. Moreover, the kernel of Ψ is the ideal $\mathscr{I}^\infty$ of ∞-phantom maps, hence Ψ induces an equivalence of categories $\mathscr{T}/\mathscr{I}^\infty \simeq \mathbf{Pres}_\infty^{\simeq}$.

We prove Theorem 6.14 at the end of this section.

Lemma 6.15. *Given a Postnikov resolution (X, X_*, P_*), the following sequence is exact for $n \geq 0$,*

$$S_\alpha(P_{n+1}) \xrightarrow{S_\alpha(f_{n+1})} S_\alpha(X_n) \xrightarrow[+1]{S_\alpha(p_n)} \mathrel{\mkern-14mu}\rightarrow S_\alpha(X).$$

Moreover, $S_\alpha(p_n)$ splits for $n > 0$.

Proof. For $n = 0$ it holds by definition since q_0 is an isomorphism. For $n > 0$, consider an associated Adams resolution via Lemma 6.10. Since $j_n \cdots j_0$ and j_{n+1} are phantoms

$$\begin{array}{c} S_\alpha(W_n) \hookrightarrow^{S_\alpha(\phi_n)} S_\alpha(X_n) \xrightarrow[+1]{S_\alpha(p_n)} \mathrel{\mkern-14mu}\rightarrow S_\alpha(X) \\ S_\alpha(W_{n+1}) \overset{S_\alpha(r_{n+1})}{\underset{+1}{\hookrightarrow}} S_\alpha(P_{n+1}) \xrightarrow{S_\alpha(g_{n+1})} \mathrel{\mkern-14mu}\rightarrow S_\alpha(W_n) \end{array}$$

are short exact by Lemma 6.2, and $f_{n+1} = \phi_n g_{n+1}$, hence the sequence in the statement is exact.

Now let $n > 0$. Recall that the sequence

$$0 \longleftarrow S_\alpha(X) \xleftarrow{S_\alpha(p_0 q_0^{-1})} S_\alpha(P_0) \xleftarrow[+1]{S_\alpha(q_0 f_1)} S_\alpha(P_1).$$

is exact. The map $S_\alpha(i_n \cdots i_1 q_0^{-1}) \colon S_\alpha(P_0) \stackrel{-1}{\to} S_\alpha(X_n)$ factors uniquely through $S_\alpha(p_0 q_0^{-1}) \colon S_\alpha(P_0) \twoheadrightarrow S_\alpha(X)$ since $(i_n \cdots i_1 q_0^{-1})(q_0 f_1) = i_n \cdots i_1 f_1 = 0$. The factorization $S_\alpha(X) \stackrel{-1}{\to} S_\alpha(X_n)$ composed with $S_\alpha(p_n)$ is the identity in $S_\alpha(X)$ since $p_n(i_n \cdots i_1 q_0^{-1}) = p_0 q_0^{-1}$, hence we are done. $\square$

Proposition 6.16. *Given a morphism $h \colon X \to Y$ in $\mathscr{T}$ and Postnikov resolutions (X, X_*, P_*) and (Y, Y_*, Q_*) there exists a morphism of Postnikov resolutions as in* (6.13) *extending h.*

Proof. We proceed by induction. The morphisms φ_0 and φ_1 can be constructed by completing the following diagram of exact rows

$$\begin{tikzcd}
S_\alpha(P_1) \arrow[r, "S_\alpha(q_0 f_1)", "+1"'] & S_\alpha(P_0) \arrow[r, two heads, "S_\alpha(p_0 q_0^{-1})"] & S_\alpha(X) \arrow[d, "S_\alpha(h)"] \\
S_\alpha(Q_1) \arrow[r, "S_\alpha(q_0 f_1)", "+1"'] & S_\alpha(Q_0) \arrow[r, two heads, "S_\alpha(p_0 q_0^{-1})"] & S_\alpha(Y)
\end{tikzcd}$$

to commutative squares, and $\psi_0 = (q_0^Y)^{-1}\varphi_0 q_0^X$.

Assume we have constructed up to the following diagram of solid arrows

$$\begin{tikzcd}
X \arrow[ddd, "h"'] & & & & & & & \\
& \cdots & X_{n-2} \arrow[rr] \arrow[ull, bend right] \arrow[ddd, "\psi_{n-2}"'] & & X_{n-1} \arrow[rr] \arrow[dl] \arrow[ullll, bend right] \arrow[ddd, "\psi_{n-1}"'] & & X_n \arrow[dl] \arrow[ullllll, bend right] \arrow[ddd, dashed, "\psi'_n"] & \cdots \\
& & & P_{n-1} \arrow[ul] \arrow[ddd, "\varphi_{n-1}"] & & P_n \arrow[ul] \arrow[ddd, "\varphi_n"] & & \\
Y & & & & & & & \\
& \cdots & Y_{n-2} \arrow[rr] \arrow[ul, bend right] & & Y_{n-1} \arrow[rr] \arrow[dl] \arrow[ulll, bend right] & & Y_n \arrow[dl] \arrow[ulllll, bend right] & \cdots \\
& & & Q_{n-1} \arrow[ul] & & Q_n \arrow[ul] & &
\end{tikzcd}$$

for some $n > 0$. We extend ψ_{n-1} and φ_n to a morphism of exact triangles. In general,

(6.17)
$$\begin{tikzcd}
X_n \arrow[r, "p_n^X", "+1"'] \arrow[d, "\psi'_n"'] & X \arrow[d, "h"] \\
Y_n \arrow[r, "p_n^Y", "+1"'] & Y
\end{tikzcd}$$

does not commute, but precomposing with $i_n^X \colon X_{n-1} \to X_n$,

$$p_n^Y \psi'_n i_n^X = p_n^Y i_n^Y \psi_{n-1} = p_{n-1}^Y \psi_{n-1} = h p_{n-1}^X = h p_n^X i_n^X.$$

Hence, $h p_n^X - p_n^Y \psi'_n$ factors as

$$X_n \xrightarrow[+1]{q_n^X} P_n \xrightarrow{\beta} Y.$$

The composite

$$S_\alpha(P_n) \xrightarrow{S_\alpha(\beta)} S_\alpha(Y) \xhookrightarrow[-1]{\substack{\text{splitting in} \\ \text{the proof of} \\ \text{Lemma 6.15}}} S_\alpha(Y_n),$$

is the image by S_α of a unique $\gamma \colon P_n \stackrel{-1}{\to} Y_n$ since $S_\alpha(P_n)$ is projective. This morphism satisfies $p_n^Y \gamma = \beta$ and $q_n^Y \gamma = 0$. The first equation holds by the splitting condition. For the second equation it is enough to check that $S_\alpha(q_n^Y \gamma) = 0$, and this holds since the splitting of $S_\alpha(p_n)$ in Lemma 6.15 is induced by $S_\alpha(i_n \cdots i_1 q_0^{-1})$

and $q_n^Y i_n^Y = 0$. Hence, the morphism $\psi_n = \psi_n' + \gamma q_n^X$ still extends ψ_{n-1} and φ_n to a morphism of exact triangles since

$$q_n^Y \psi_n = q_n^Y \psi_n' + q_n^Y \gamma q_n^X = \varphi_n q_n^X + 0 q_n^X = \varphi_n q_n^X,$$
$$\psi_n i_n^X = \psi_n' i_n^X + \gamma q_n^X i_n^X = i_n^Y \psi_n + \gamma 0 = i_n^Y \psi_n.$$

Moreover, the square (6.17) commutes if we replace ψ_n' with ψ_n since

$$p_n^Y \psi_n = p_n^Y \psi_n' + p_n^Y \gamma q_n^X = p_n^Y \psi_n' + \beta q_n^X = p_n^Y \psi_n' + h p_n^X - p_n^Y \psi_n' = h p_n^X.$$

In order to conclude the induction step we must take $\varphi_{n+1} \colon P_{n+1} \to Q_{n+1}$ completing

$$\begin{tikzcd}
P_{n+1} \arrow[r, "f_{n+1}^X"] & X_n \arrow[r, "p_n^X", "+1"'] \arrow[d, "\psi_n"] & X \arrow[d, "h"] \\
Q_{n+1} \arrow[r, "f_{n+1}^Y"] & Y_n \arrow[r, "p_n^Y", "+1"'] & Y
\end{tikzcd}$$

to a commutative square. This can be done. Actually, by Lemma 6.15, it is enough to notice that that $p_n^Y \psi_n f_{n+1}^X = h p_{n+1}^X f_{n+1}^X = h0 = 0$. □

Proposition 6.18. *If (X, X_*, P_*) is a Postnikov resolution, then $h \colon X \to Y$ is an n-phantom map, $n > 0$, if and only if $hp_{n-1} = 0$. In particular, h is an ∞-phantom map if and only if $hp_n = 0$ for all $n \geq 0$.*

Proof. Since X_{n-1} is n-cellular, if h is an n-phantom map, then $hp_{n-1} = 0$, see Proposition 6.5. Conversely, by Lemma 6.10 the morphism p_{n-1} fits in an exact triangle

$$\begin{tikzcd}
X_{n-1} \arrow[rr, "p_{n-1}", "+1"'] & & X \arrow[dl, "j_{n-1} \cdots j_0"] \\
& W_{n-1} \arrow[ul, "\phi_{n-1}"] &
\end{tikzcd}$$

with $j_{n-1} \cdots j_0$ an n-phantom map. Therefore, if $hp_{n-1} = 0$ then h factors through $j_{n-1} \cdots j_0$, so h is an n-phantom map too. □

Proposition 6.19. *A morphisms of Postnikov resolutions as in* (6.13) *is nullhomotopic if and only if h is an ∞-phantom map.*

Proof. If we assume that (h, ψ_*, φ_*) is nullhomotopic, then ψ_n factors through f_{n+1}^Y for all $n \geq 0$. By Lemma 6.15, $p_n^Y f_{n+1}^Y = 0$ and then, $hp_n^X = p_n^Y \psi_n = 0$. Hence, h is an ∞-phantom map by Corollary 6.18.

Assume now that h is an ∞-phantom map. We construct by induction on $n \geq 0$ a map $\beta_n \colon P_n \overset{-1}{\to} Q_{n+1}$ such that the following square commutes

$$\begin{tikzcd}
P_n \arrow[r, "q_n^X", "+1"'] \arrow[d, "\psi_n"'] & X_n \arrow[d, "-1"', "\beta_n"] \\
Y_n & Q_{n+1}. \arrow[l, "f_{n+1}^Y"]
\end{tikzcd}$$

For $n=0$, the following diagram with exact rows

$$\begin{tikzcd}
S_\alpha(P_1) \arrow[r, "S_\alpha(q_0f_1)", "+1"'] \arrow[d, "S_\alpha(\varphi_1)"] & S_\alpha(P_0) \arrow[r, two heads, "S_\alpha(p_0q_0^{-1})"] \arrow[d, "S_\alpha(\varphi_0)"] & S_\alpha(X) \arrow[d, "S_\alpha(h)=0"] \\
S_\alpha(Q_1) \arrow[r, "S_\alpha(q_0f_1)", "+1"'] & S_\alpha(Q_0) \arrow[r, two heads, "S_\alpha(p_0q_0^{-1})"] & S_\alpha(Y)
\end{tikzcd}$$

shows that we can take $\beta_0\colon P_0 \to Q_1$ with $\varphi_0 = q_0^Y f_1^Y \beta_0$. This choice of β_0 works since $\psi_0 = (q_0^Y)^{-1}\varphi_0 q_0^X$.

Assume we have checked our claim up to $n-1$. Choose an Adams resolution (Y, W_*, Q_*) associated to the Postnikov resolution (Y, Y_*, Q_*) in the sense of Lemma 6.10. We use the notation therein, exchanging X and P with Y and Q, respectively. Since h is an ∞-phantom map, by Corollary 6.18,

$$p_n^Y \psi_n = h p_n^X = 0,$$

so ψ_n factors as $X_n \overset{\gamma_n}{\to} W_n \overset{\phi_n}{\to} Y_n$. By induction hypothesis,

$$\phi_n \gamma_n i_n^X = \psi_n i_n^X = i_n^Y \psi_{n-1} = i_n^Y f_n^Y \beta_{n-1} q_{n-1}^X = 0\beta_{n-1} q_{n-1}^X = 0.$$

Since $j_n \cdots j_0$ is an $(n+1)$-phantom map and X_{n-1}, the source of i_n^X, is n-cellular, the homomorphism

$$\mathscr{T}(X_{n-1}, \phi_n)\colon \mathscr{T}(X_{n-1}, W_n) \longrightarrow \mathscr{T}(X_{n-1}, Y_n)$$

is injective, so the previous equation yields $\gamma_n i_n^X = 0$. Hence, γ_n factors as

$$X_n \xrightarrow[+1]{q_n^X} P_n \xrightarrow[-1]{\varepsilon_n} W_n.$$

Furthermore, since j_{n+1} is a phantom map, $S_\alpha(g_{n+1})\colon S_\alpha(Q_{n+1}) \to S_\alpha(W_n)$ is an epimorphism and we can factor ε_n as

$$P_n \xrightarrow[-1]{\beta_n} Q_{n+1} \xrightarrow{g_{n+1}} W_n.$$

Finally, $f_{n+1}^Y \beta_n q_n^X = \phi_n g_{n+1} \beta_n q_n^X = \phi_n \varepsilon_n q_n^X = \phi_n \gamma_n = \psi_n$. □

Proof of Theorem 6.14. Any object X in $\mathscr{T}$ has a Postnikov resolution $\Psi(X)$ by Corollary 6.11. We choose one. Proposition 6.16 proves that there are choices for $\Psi(h)$ as in the statement. Moreover, the choice is unique in the homotopy category by Proposition 6.19. By uniqueness, Ψ must be an additive functor. Propositions 6.16 and 6.19 prove that any two Postnikov resolutions of X are isomorphic in $\mathbf{Pres}_\infty^{\simeq}$, hence Ψ is essentially surjective. Moreover, Ψ is full since the homotopy class of an arbitrary morphism $(h, \psi_*, \varphi_*)\colon \Psi(X) \to \Psi(Y)$ is $\Psi(h)$. Finally, the kernel of Ψ is $\mathscr{I}^\infty$ by Proposition 6.19. □

6.4. **Homotopy colimits and Postnikov resolutions.** Recall that a *homotopy colimit* [Nee01b, Definition 1.6.4] of a sequence in a triangulated category with countable coproducts $\mathscr{T}$

$$X_0 \xrightarrow{i_1} X_1 \xrightarrow{i_2} X_2 \xrightarrow{i_3} X_3 \longrightarrow \cdots$$

is an exact triangle

$$\begin{tikzcd} \coprod_{n>0} X_n \arrow[rr, "(6.21)"] & & \coprod_{n>0} X_n \arrow[dl, "(p'_n)_{n>0}"', "+1"] \\ & \operatorname{Hocolim}_n X_n \arrow[ul, "\delta'"] & \end{tikzcd} \tag{6.20}$$

where the upper arrow is given by the following matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -i_2 & 1 & 0 & 0 & \\ 0 & -i_3 & 1 & 0 & \\ 0 & 0 & -i_4 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}. \tag{6.21}$$

Usually, δ' is taken to be the degree $+1$ map, but the previous convention is more convenient for our purposes. Moreover, X_0 and i_1 are usually not neglected in (6.20) and (6.21), but the construction turns out to be equivalent, see [Nee01b, Lemma 1.7.1].

Proposition 6.22. *Given a Postnikov resolution (X, X_*, P_*), there is a homotopy colimit given by an exact triangle of the form*

$$\begin{tikzcd} \coprod_{n>0} X_n \arrow[rr, "(6.21)"] & & \coprod_{n>0} X_n \arrow[dl, "(p_n)_{n>0}"', "+1"] \\ & X. \arrow[ul, "\delta"] & \end{tikzcd}$$

In the proof of Proposition 6.22 we use the following lemma.

Lemma 6.23. *Given morphisms*

$$X \xrightarrow{f} Y \xrightarrow[+1]{i} Z$$

such that $if = 0$ and

$$S_\alpha(X) \overset{S_\alpha(f)}{\hookrightarrow} S_\alpha(Y) \xrightarrow[+1]{S_\alpha(i)} S_\alpha(Z)$$

is a short exact sequence, there is an exact triangle

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \arrow[dl, "i"', "+1"] \\ & Z. \arrow[ul, "q"] & \end{tikzcd}$$

Proof. Complete f to an exact triangle

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \arrow[dl, "i'"', "+1"] \\ & Z'. \arrow[ul, "q'"] & \end{tikzcd}$$

Since $if = 0$, i factors as $Y \overset{i'}{\to} Z' \overset{\phi}{\to} Z$. Since $S_\alpha(f)$ is a monomorphism, the sequence

$$S_\alpha(X) \xhookrightarrow{S_\alpha(f)} S_\alpha(Y) \underset{+1}{\xrightarrow{S_\alpha(i')}} S_\alpha(Z')$$

is also short exact by Lemma 6.2. Therefore, $S_\alpha(\phi)$ is an isomorphism. Finally, since S_α reflects isomorphisms, ϕ is an isomorphism and we can take $q = q'\phi^{-1}$. □

Proof of Proposition 6.22. Clearly, $(p_n)_{n>0}(6.21) = 0$ since $p_n = p_{n+1}i_{n+1}$, $n > 0$. Using the splitting $S_\alpha(X_n) \cong S_\alpha(X) \oplus \operatorname{Im} S_\alpha(f_{n+1})$ given by Lemma 6.15, $n > 0$, and the fact that S_α preserves coproducts, we can identify $S_\alpha(6.21)$ with the endomorphism of

$$\left(\bigoplus_{n>0} S_\alpha(X)\right) \oplus \left(\bigoplus_{n>0} \operatorname{Im} S_\alpha(f_{n+1})\right) \tag{6.24}$$

which decomposes as the identity on the second factor, since $i_n f_n = 0$, and the endomorphism defined by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \\ 0 & -1 & 1 & 0 & \\ 0 & 0 & -1 & 1 & \\ \vdots & & & & \ddots \end{pmatrix} \tag{6.25}$$

on the first factor.

The endomorphism (6.25), and hence $S_\alpha(6.21)$, is a split monomorphism. The matrix

$$\begin{pmatrix} 0 & -1 & -1 & -1 & \cdots \\ 0 & 0 & -1 & -1 & \\ 0 & 0 & 0 & -1 & \\ 0 & 0 & 0 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$$

defines a retraction of (6.25). The cokernel of (6.25) $\coprod$ id is $S_\alpha(X)$. The natural projection is 0 on the second factor of (6.24) and

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \end{pmatrix}$$

on the first factor. This morphism identifies with $S_\alpha(p_n)_{n>0}$ via the direct sum decomposition, since $p_n f_{n+1} = 0$ by Lemma 6.15. Therefore, Lemma 6.23 applies. □

The following corollary is a new result. It should be compared to the fact that, if $\aleph_0$-Adams representability holds then the ideal of phantom maps is a square zero ideal, cf. [Nee97]. Actually, one can check along the same lines that this is also true under α-Adams representability.

Corollary 6.26. *The ideal $\mathscr{I}^\infty$ of infinite phantom maps is a square zero ideal $(\mathscr{I}^\infty)^2 = 0$, i.e. if $h\colon X \to Y$ and $k\colon Y \to Z$ are ∞-phantom maps, then $kh = 0$.*

Proof. Consider a homotopy colimit as in the statement of Proposition 6.22. Since h is an ∞-phantom, $0 = (hp_n)_{n>0} = h(p_n)_{n>0}$ by Proposition 6.18, hence h factors as

$$X \xrightarrow{\delta} \coprod_{n>0} X_n \xrightarrow{h'} Y.$$

Since k is an ∞-phantom map and each X_n is n-cellular, $n > 0$, $kh' = 0$ by Proposition 6.5. Finally, $kh = kh'\delta = 0\delta = 0$. $\square$

Remark 6.27. Theorem 6.14 and Corollary 6.26 show that

$$\mathscr{I}^{\infty} \rightarrowtail \mathscr{T} \overset{\Psi}{\twoheadrightarrow} \mathbf{Pres}_{\infty}^{\simeq}$$

is a weak linear extension [Bau91, Definition II.1.7], therefore the $\mathscr{T}$ bimodule $\mathscr{I}^{\infty}$ is actually a $\mathbf{Pres}_{\infty}^{\simeq}$-bimodule and the weak linear extension is classified up to equivalence by a class in cohomology of categories

$$\{\mathscr{T}\} \in H^2(\mathbf{Pres}_{\infty}^{\simeq}, \mathscr{I}^{\infty}).$$

This can be compared to the fact, under $\aleph_0$-Adams representability (and also under α-Adams representability replacing $\aleph_0$ with α, as one can easily deduce from the results of this paper) $\mathscr{T}$ is a linear extension of the full subcategory of $\aleph_0$-flat objects in $\mathrm{Mod}_{\aleph_0}(\mathscr{T}^{\aleph_0})$ by $\mathscr{I}$, cf. [CS98, §5].

6.5. **Postnikov systems.** Postnikov systems were introduced in [BKS04]. In this section we make them the objects of a certain category where we define a natural homotopy relation. The main result of this section establishes an equivalence between the homotopy category of Postnikov resolutions, defined in Section 6.3, and the homotopy category of Postnikov systems.

Definition 6.28. A *Postnikov system* (X_*, P_*) is a countable sequence of exact triangles

$$\begin{tikzcd}
0 \arrow[rr, "i_0"] & & X_0 \arrow[dl, "q_0"', "+1"] \arrow[rr, "i_1"] & & X_1 \arrow[dl, "q_1"', "+1"] \arrow[rr, "i_2"] & & X_2 \arrow[dl, "q_2"', "+1"] \arrow[rr, "i_3"] & & X_3 \arrow[dl, "q_3"', "+1"] & \cdots \\
& P_0 \arrow[ul, "f_0"] & & P_1 \arrow[ul, "f_1"] & & P_2 \arrow[ul, "f_2"] & & P_3 \arrow[ul, "f_3"] & &
\end{tikzcd}$$

such that S_α maps

$$P_0 \xleftarrow[q_0 f_1]{+1} P_1 \xleftarrow[q_1 f_2]{+1} P_2 \xleftarrow[q_2 f_3]{+1} P_3 \longleftarrow \cdots.$$

to an exact sequence of projective objects in $\mathrm{Mod}_\alpha(\mathscr{C})$. In particular, X_n is $(n+1)$-cellular. We will denote the structure morphisms by f_n^X, i_n^X and q_n^X when we need to distinguish between different Postnikov systems.

A *morphism of Potsnikov systems*

$$(\psi_*, \varphi_*) \colon (X_*, P_*) \longrightarrow (Y_*, Q_*)$$

is a sequence of exact triangle morphisms as follows

$$\begin{tikzcd}
0 \arrow[rr] \arrow[dd] & & X_0 \arrow[rr] \arrow[dl, "+1"] \arrow[dd, "\psi_0"] & & X_1 \arrow[rr] \arrow[dl, "+1"] \arrow[dd, "\psi_1"] & & X_2 \arrow[rr] \arrow[dl, "+1"] \arrow[dd, "\psi_2"] & & X_3 \arrow[dl, "+1"] \arrow[dd, "\psi_3"] & \\
& P_0 \arrow[ul] \arrow[dd, "\varphi_0"] & & P_1 \arrow[ul] \arrow[dd, "\varphi_1"] & & P_2 \arrow[ul] \arrow[dd, "\varphi_2"] & & P_3 \arrow[ul] \arrow[dd, "\varphi_3"] & & \cdots \\
0 \arrow[rr] & & Y_0 \arrow[rr] \arrow[dl, "+1"] & & Y_1 \arrow[rr] \arrow[dl, "+1"] & & Y_2 \arrow[rr] \arrow[dl, "+1"] & & Y_3 \arrow[dl, "+1"] & \\
& Q_0 \arrow[ul] & & Q_1 \arrow[ul] & & Q_2 \arrow[ul] & & Q_3 \arrow[ul] & &
\end{tikzcd}$$

Composition of morphisms of Postnikov systems is defined in the obvious way.

A pair of morphisms

$$(\psi_*, \varphi_*), (\bar{\psi}_*, \bar{\varphi}_*) \colon (X_*, P_*) \longrightarrow (Y_*, Q_*)$$

are *homotopic* $(\psi_*, \varphi_*) \simeq (\bar{\psi}_*, \bar{\varphi}_*)$ if the four equivalent conditions (1–4) in Definition 6.12 are satisfied. This natural equivalence relation is additive: two morphisms are homotopic iff their difference $(\psi_* - \bar{\psi}_*, \varphi_* - \bar{\varphi}_*)$ is nullhomotopic. We denote $\mathbf{Post}_\infty$ the category of Postnikov systems and $\mathbf{Post}_\infty^{\simeq}$ its homotopy category. Both of them are additive.

Theorem 6.29. *The forgetful functor*

$$\Phi \colon \mathbf{Pres}_\infty^{\simeq} \longrightarrow \mathbf{Post}_\infty^{\simeq}, \quad \Phi(X, X_*, P_*) = (X_*, P_*),$$

is an equivalence of categories surjective on objects.

This theorem is proved after the following lemma.

Lemma 6.30. *In a Postnikov system (X_*, P_*), $S_\alpha(i_1 q_0^{-1})$ induces a degree -1 isomorphism $H_0 S_\alpha(P_*) \cong \operatorname{Im} S_\alpha(i_1)$, and $S_\alpha(i_n)$ induces a degree 0 isomorphism $\operatorname{Im} S_\alpha(i_n) \cong \operatorname{Im} S_\alpha(i_{n+1})$, $n > 0$. In particular, $S_\alpha(X_n) \cong H_0 S_\alpha(P_*) \oplus \operatorname{Ker} S_\alpha(i_{n+1})$ for $n > 0$.*

Proof. The functor S_α takes exact triangles to exact sequences, therefore

$$\begin{tikzcd}
S_\alpha(X_*) \arrow[rr, "S_\alpha(i_*)", "(+1,0)"'] & & S_\alpha(X_*) \arrow[dl, "S_\alpha(q_*)"', "(0,+1)"] \\
& S_\alpha(P_*) \arrow[ul, "(-1,0)"', "S_\alpha(f_*)"] &
\end{tikzcd}$$

is an exact couple. Here the first degree corresponds to the subscript $*$, and the second degree is the internal degree in the graded abelian category $\operatorname{Mod}_\alpha(\mathscr{C})$.

Since $S_\alpha(P_*)$ is exact in degrees $\neq 0$, the derived exact couple is

$$\begin{tikzcd}
\operatorname{Im} S_\alpha(i_*) \arrow[rr, "(+1,0)"'] & & \operatorname{Im} S_\alpha(i_*) \arrow[dl, "(-1,+1)"] \\
& H_0 S_\alpha(P_*) \arrow[ul, "(0,0)"', "0"] &
\end{tikzcd}$$

with $H_0 S_\alpha(P_*)$ concentrated in degree 0. Indeed, since $\operatorname{Im} S_\alpha(i_*)$ is concentrated in degrees > 0, the map $H_0 S_\alpha(P_*) \to \operatorname{Im} S_\alpha(i_*)$ is the trivial morphism, hence the lemma follows. $\square$

Proof of Theorem 6.29. Let (X_*, P_*) be a Postnikov system. Take a homotopy colimit as in (6.20). We claim that $(\mathrm{Hocolim}_n X_n, X_*, P_*)$ is a Postnikov resolution. Actually, it is only left to check that $S_\alpha(\mathrm{Hocolim}_n X_n) = H_0 S_\alpha(P_*)$. By Lemma 6.30, $S_\alpha(6.21)$ can be identified with the endomorphism of

$$\left(\bigoplus_{n>0} H_0 S_\alpha(P_*)\right) \oplus \left(\bigoplus_{n>0} \operatorname{Ker} S_\alpha(i_{n+1})\right) \tag{6.31}$$

which decomposes as the identity on the second factor and (6.25) on the first factor, compare the proof of Proposition 6.22. Proceeding as in that proof, we deduce that the cokernel of $S_\alpha(6.21)$ is $H_0 S_\alpha(P_*)$. This cokernel can also be identified with $S_\alpha(\mathrm{Hocolim}_n X_n)$ by Lemma 6.2. This proves the claim and that Φ is surjective on objects.

Let (X, X_*, P_*) and (Y, Y_*, Q_*) be Postnikov resolutions and $(\psi_*, \varphi_*)\colon (X_*, P_*) \to (Y_*, Q_*)$ a morphism of Postnikov systems. We choose exact triangles defining homotopy colimits as in Proposition 6.22. The following commutative square of solid arrows can be extended to a triangle morphism

$$\begin{tikzcd}
\coprod_{n>0} X_n \arrow[rr, "(6.21)"] \arrow[dd, "(\psi_n)_{n>0}"'] & & \coprod_{n>0} X_n \arrow[dl, "(p_n)_{n>0}"', "+1"] \arrow[dd, "(\psi_n)_{n>0}"] \\
& X \arrow[ul] \arrow[dd, dashed, "h"] & \\
\coprod_{n>0} Y_n \arrow[rr, "(6.21)"] & & \coprod_{n>0} Y_n \arrow[dl, "+1"', "(p_n)_{n>0}"] \\
& Y \arrow[ul] &
\end{tikzcd}$$

Hence, $(h, \psi_*, \varphi_*)\colon (X, X_*, P_*) \to (Y, Y_*, Q_*)$ is a morphism of Postnikov resolutions. This shows that Φ is full.

The functor Φ is faithful since two morphisms of Postnikov resolutions are homotopic if and only if the underlying morphisms of Postnikov systems are. $\square$

Remark 6.32. By Theorem 6.29 and Remark 6.27,

$$\mathscr{I}^\infty \rightarrowtail \mathscr{T} \overset{\Phi\Psi}{\twoheadrightarrow} \mathbf{Post}_\infty^{\simeq}$$

is a weak linear extension, the $\mathscr{T}$ bimodule $\mathscr{I}^\infty$ is actually a $\mathbf{Post}_\infty^{\simeq}$-bimodule and the weak linear extension is classified up to equivalence by a class in cohomology of categories

$$\{\mathscr{T}\} \in H^2(\mathbf{Post}_\infty^{\simeq}, \mathscr{I}^\infty).$$

It is interesting to notice that $\mathbf{Post}_\infty^{\simeq}$ only depends of the full subcategory of cellular objects in $\mathscr{T}$, and that there are no non-trivial ∞-phantom maps between two cellular objects. Hence, the previous linear extension is a way of breaking $\mathscr{T}$ into an ∞-phantom part and an ∞-phantomless part.

6.6. **Truncated Postnikov systems and obstructions.** Our notion of truncated Potsnikov system enriches that considered in [BKS04] in a way which is suitable to develop an obstruction theory. We also define homotopy categories of truncated Potsnikov systems.

Definition 6.33. An *n-truncated Postnikov system* $(X_{\leq n}, P_*)$, $n \geq 0$, is a diagram in $\mathscr{T}$

$$\begin{tikzcd}
0 \arrow[rr, "i_0"] & & X_0 \arrow[rr, "i_1"] \arrow[dl, "q_0", "+1"'] & & X_1 \arrow[dl, "q_1", "+1"'] & \cdots\cdots & X_{n-1} \arrow[rr, "i_n"] & & X_n \arrow[dl, "q_n", "+1"'] & & & \\
& P_0 \arrow[ul, "f_0"] & & P_1 \arrow[ul, "f_1"] & & & & P_n \arrow[ul, "f_n"] & & P_{n+1} \arrow[ul, "f_{n+1}"'] & P_{n+2} \arrow[l, "d_{n+2}"', "+1"] & \cdots \arrow[l]
\end{tikzcd}$$

where the first $n+1$ triangles are exact, the *cocycle condition*

$$f_{n+1}d_{n+2} = 0$$

is satisfied, and the restricted Yoneda functor maps

$$P_0 \xleftarrow[q_0 f_1]{+1} P_1 \longleftarrow \cdots\cdots \longleftarrow P_n \xleftarrow[q_n f_{n+1}]{+1} P_{n+1} \xleftarrow[d_{n+2}]{+1} P_{n+2} \longleftarrow \cdots$$

to an exact sequence of projective objects. For $0 \leq k \leq n$ we denote

$$d_{k+1} = q_k f_{k+1}.$$

Notice that X_k is $(n+1)$-cellular, $0 \leq k \leq n$. We will denote the structure morphisms by f_k^X, $0 \leq k \leq n+1$, i_k^X, q_k^X, $0 \leq k \leq n$, and d_k^X, $k > 0$, if we need to distinguish between different n-truncated Postnikov systems.

A *morphism of n-truncated Potsnikov systems*

$$(\psi_{\leq n}, \varphi_*) \colon (X_{\leq n}, P_*) \longrightarrow (Y_{\leq n}, Q_*)$$

is a diagram

$$\begin{tikzcd}
0 \arrow[rr] \arrow[dd] & & X_0 \arrow[rr] \arrow[dl] \arrow[dd, "\psi_0"] & & X_1 \arrow[dl] \arrow[dd, "\psi_1"] & \cdots\cdots & X_{n-1} \arrow[rr] \arrow[dd, "\psi_{n-1}"] & & X_n \arrow[dl] \arrow[dd, "\psi_n"] & & & \\
& P_0 \arrow[ul] \arrow[dd, "\varphi_0"] & & P_1 \arrow[ul] \arrow[dd, "\varphi_1"] & & & & P_n \arrow[ul] \arrow[dd, "\varphi_n"] & & P_{n+1} \arrow[ul] \arrow[dd, "\varphi_{n+1}"] & P_{n+2} \arrow[l] \arrow[dd, "\varphi_{n+2}"] & \cdots \arrow[l] \\
0 \arrow[rr, crossing over] & & Y_0 \arrow[rr, crossing over] \arrow[dl] & & Y_1 \arrow[dl] & & Y_{n-1} \arrow[rr, crossing over] & & Y_n \arrow[dl] & & & \\
& Q_0 \arrow[ul] & & Q_1 \arrow[ul] & & & & Q_n \arrow[ul] & & Q_{n+1} \arrow[ul] & Q_{n+2} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

where all squares commute. Composition is defined in the obvious way.

A pair of morphisms of n-truncated Postnikov systems

$$(\psi_{\leq n}, \varphi_*), (\bar{\psi}_{\leq n}, \bar{\varphi}_*) \colon (X_{\leq n}, P_*) \longrightarrow (Y_{\leq n}, Q_*)$$

are *homotopic* $(\psi_{\leq n}, \varphi_*) \simeq (\bar{\psi}_{\leq n}, \bar{\varphi}_*)$ if $\psi_k - \bar{\psi}_k$ factors through $f_{k+1} \colon Q_{k+1} \to Y_k$ for $0 \leq k \leq n$. This condition can be characterized in different ways for $k < n$, see Definition 6.12 (1–4).

The homotopy natural equivalence relation is additive: two morphisms are homotopic iff their difference $(\psi_{\leq n} - \bar{\psi}_{\leq n}, \varphi_* - \bar{\varphi}_*)$ is nullhomotopic. We denote $\mathbf{Post}_n$ the category of n-truncated Postnikov systems and $\mathbf{Post}_n^{\simeq}$ its homotopy category. Both categories are additive and the natural projection $\mathbf{Post}_n \to \mathbf{Post}_n^{\simeq}$ is an additive functor.

The *homology functor*

$$\begin{aligned}\mathbf{Post}_n &\longrightarrow \mathrm{Mod}_\alpha(\mathscr{C}),\\ (X_{\le n}, P_*) &\mapsto H_0 S_\alpha(P_*) = \operatorname{Coker} S_\alpha(d_1),\end{aligned}$$

factors through the homotopy category,

$$\mathbf{Post}_n^{\simeq} \longrightarrow \mathrm{Mod}_\alpha(\mathscr{C}).$$

This factorization is an equivalence for $n = 0$.

The *n-truncation functor*, $n > 0$,

$$t_{n-1}\colon \mathbf{Post}_n \longrightarrow \mathbf{Post}_{n-1}$$

is the functor $t_{n-1}(X_{\le n}, P_*) = (X_{\le n-1}, P_*)$ defined by forgetting X_n, f_{n+1}, i_n, and q_n, but not $d_{n+1} = f_{n+1}q_n$. This functor is additive and compatible with the homotopy relation, hence it induces an additive functor

$$t_{n-1}\colon \mathbf{Post}_n^{\simeq} \longrightarrow \mathbf{Post}_{n-1}^{\simeq}.$$

Lemma 6.34. *Given an n-truncated Postnikov system $(X_{\le n}, P_*)$:*

- *$S_\alpha(i_1 q_0^{-1})$ induces a degree -1 isomorphism $H_0S_\alpha(P_*) \cong \operatorname{Im} S_\alpha(i_1)$,*
- *$S_\alpha(i_{k+1})$ induces a degree 0 isomorphism $\operatorname{Im} S_\alpha(i_k) \cong \operatorname{Im} S_\alpha(i_{k+1})$ for $0 < k < n$,*
- *the natural projection $S_\alpha(X_k) \twoheadrightarrow \operatorname{Coker} S_\alpha(f_{k+1})$ restricts to a degree 0 isomorphism $\operatorname{Im} S_\alpha(i_k) \cong \operatorname{Coker} S_\alpha(f_{k+1})$, for $0 < k \le n$.*

In particular, for $0 < k \le n$, $S_\alpha(X_k) \cong H_0S_\alpha(P_*) \oplus \operatorname{Im} S_\alpha(f_{k+1})$.

Proof. Extend f_{n+1} to an exact triangle,

$$\begin{tikzcd}
\cdots & X_{n-1} \arrow[rr, "i_n"] & & X_n \arrow[dl, "+1"', "q_n"] \arrow[rr, dashed, "i_{n+1}"] & & X_{n+1} \arrow[dl, dashed, "+1"', "q_{n+1}"] \\
& & P_n \arrow[ul, "f_n"] & & P_{n+1}. \arrow[ul, "f_{n+1}"] &
\end{tikzcd}$$

Consider the following exact couple in $\mathrm{Mod}_\alpha(\mathscr{C})$,

$$\begin{tikzcd}
S_\alpha(X_*) \arrow[rr, "S_\alpha(i_*)", "(+1,0)"'] & & S_\alpha(X_*) \arrow[dl, "(0,+1)"', "S_\alpha(q_*)"] \\
& S_\alpha(P_*). \arrow[ul, "(-1,0)"', "S_\alpha(f_*)"] &
\end{tikzcd}$$

Here for $k > n+1$ we set $X_k = X_{n+1}$, $P_k = 0$ and $i_k = \mathrm{id}_{X_{n+1}}$. The E^2-term of the induced spectral sequence is

$$E_0^2 = \operatorname{Coker} S_\alpha(d_1) = H_0S_\alpha(P_*), \qquad E_{n+1}^2 = \operatorname{Ker} S_\alpha(d_{n+1}),$$

and $E_k^2 = 0$ otherwise. The derived exact couple is

$$\begin{tikzcd}
\operatorname{Im} S_\alpha(i_*) \arrow[rr, "i'_*", "(+1,0)"'] & & \operatorname{Im} S_\alpha(i_*) \arrow[dl, "(-1,+1)"', "q'_*"] \\
& E_*^2. \arrow[ul, "(0,0)"', "f'_*"] &
\end{tikzcd}$$

Since $\operatorname{Im} S_\alpha(i_k)$ is concentrated in degrees $k > 0$, q'_* contains an isomorphism $\operatorname{Im} S_\alpha(i_1) \cong E^2_0 = H_0 S_\alpha(P_*)$ whose inverse is induced by $S_\alpha(i_1 q_0^{-1})$. By the sparsity of E^2_*, i'_* contains isomorphisms $\operatorname{Im} S_\alpha(i_k) \cong \operatorname{Im} S_\alpha(i_{k+1})$ induced by $S_\alpha(i_{k+1})$ for $0 < k \leq n$. This finishes the proof since $\operatorname{Ker} S_\alpha(i_k) = \operatorname{Im} S_\alpha(f_k)$ and hence $S_\alpha(i_k)$ induces an isomorphism $\operatorname{Coker} S_\alpha(f_k) \cong \operatorname{Im} S_\alpha(i_k)$, $0 < k \leq n+1$. □

Remark 6.35. Let $(X_{\leq n}, P_*)$ be an n-truncated Postnikov system. The following inclusion defined by Lemma 6.34, $0 < k \leq n+1$, which splits for $0 < k \leq n$, has degree -1,

$$H_0 S_\alpha(P_*) \underset{-1}{\subset} S_\alpha(X_k).$$

Notice that X_{n+1} is not part of the n-truncated Postnikov system, it is simply a mapping cone of f_{n+1}.

Definition 6.36. Let $(X_{\leq n}, P_*)$ be an n-truncated Postnikov system. Extend f_{n+1} to an exact triangle

$$\begin{tikzcd}
& X_{n-1} \arrow[rr, "i_n"] & & X_n \arrow[dl, "+1"', "q_n"] \arrow[rr, dashed, "i_{n+1}"] & & X_{n+1} \arrow[dl, dashed, "+1"', "q_{n+1}"] & & & & \\
\cdots & & P_n \arrow[ul, "f_n"] & & P_{n+1} \arrow[ul, "f_{n+1}"] & & P_{n+2} \arrow[ll, "+1", "d_{n+2}"'] \arrow[ul, dashed, "\bar{f}_{n+2}"'] & & P_{n+3} \arrow[ll, "+1", "d_{n+3}"'] & \cdots \arrow[l]
\end{tikzcd}$$

By the cocycle condition $f_{n+1} d_{n+2} = 0$ there exists $\bar{f}_{n+2}$ with $d_{n+2} = q_{n+1} \bar{f}_{n+2}$. This construction does not yield an $(n+1)$-truncated Postnikov system since $\bar{f}_{n+2} d_{n+3} \neq 0$ in general. However, $q_{n+1} \bar{f}_{n+2} d_{n+3} = d_{n+2} d_{n+3} = 0$, and then $S_\alpha(\bar{f}_{n+2} d_{n+3})$ factors through $\operatorname{Ker} S_\alpha(q_{n+1}) \cong \operatorname{Coker} S_\alpha(f_{n+1}) \cong H_0 S_\alpha(P_*)$, see Lemma 6.34, as

$$S_\alpha(\bar{f}_{n+2} d_{n+3}) \colon S_\alpha(P_{n+3}) \underset{+2}{\xrightarrow{\tilde{\kappa}}} H_0 S_\alpha(P_*) \underset{-1}{\subset} S_\alpha(X_{n+1}),$$

The morphism $\tilde{\kappa}$ satisfies $\tilde{\kappa} S_\alpha(d_{n+4}) = 0$ since $\bar{f}_{n+2} d_{n+3} d_{n+4} = 0$.

The *obstruction of an n-truncated Postnikov system* $(X_{\leq n}, P_*)$ is the element

$$\kappa(X_{\leq n}, P_*) \in \operatorname{Ext}^{n+3,-1-n}_{\alpha,\mathscr{C}}(H_0 S_\alpha(P_*), H_0 S_\alpha(P_*))$$

represented by a morphism $\tilde{\kappa}$ constructed as in the previous paragraph.

This obstruction class is natural in the following sense.

Proposition 6.37. *Given a morphism of n-truncated Postnikov systems,*

$$(\psi_{\leq n}, \varphi_*) \colon (X_{\leq n}, P_*) \longrightarrow (Y_{\leq n}, Q_*),$$

the following equation holds in $\operatorname{Ext}^{n+3,-1-n}_{\alpha,\mathscr{C}}(H_0 S_\alpha(P_*), H_0 S_\alpha(Q_*))$,

$$H_0 S_\alpha(\varphi_*) \cdot \kappa(X_{\leq n}, P_*) = \kappa(Y_{\leq n}, Q_*) \cdot H_0 S_\alpha(\varphi_*).$$

Proof. Assume we have made choices for the definition of the two obstructions. Take ψ_{n+1} extending ψ_n and φ_{n+1} to a triangle morphism,

$$\begin{tikzcd}
& X_{n-1} \arrow[rr] \arrow[dd, "\psi_{n-1}"] & & X_n \arrow[dl] \arrow[rr, dashed] \arrow[dd, "\psi_n"] & & X_{n+1} \arrow[dl, dashed] \arrow[dd, dashed, "\psi_{n+1}"] & & & \\
\cdots & & P_n \arrow[ul] \arrow[dd, "\varphi_n"] & & P_{n+1} \arrow[ul] \arrow[dd, "\varphi_{n+1}"] & & P_{n+2} \arrow[ll] \arrow[ul, dashed] \arrow[dd, "\varphi_{n+2}"] & P_{n+3} \arrow[l] \arrow[dd, "\varphi_{n+3}"] & \cdots \arrow[l] \\
& Y_{n-1} \arrow[rr] & & Y_n \arrow[dl] \arrow[rr, dashed] & & Y_{n+1} \arrow[dl, dashed] & & & \\
& & Q_n \arrow[ul] & & Q_{n+1} \arrow[ul] & & Q_{n+2} \arrow[ll] \arrow[ul, dashed] & Q_{n+3} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

The square containing ψ_{n+1} and φ_{n+2} need not commute. However,

$$\begin{aligned} q^Y_{n+1}\psi_{n+1}\bar{f}^X_{n+2} &= \varphi_{n+1}q^X_{n+1}\bar{f}^X_{n+2} = \varphi_{n+1}d^X_{n+2} \\ &= d^Y_{n+2}\varphi_{n+2} = q^Y_{n+1}\bar{f}^Y_{n+2}\varphi_{n+2}, \end{aligned}$$

hence $S_\alpha(\psi_{n+1}\bar{f}^X_{n+2} - \bar{f}^Y_{n+2}\varphi_{n+2})$ factors as

$$S_\alpha(\psi_{n+1}\bar{f}^X_{n+2} - \bar{f}^Y_{n+2}\varphi_{n+2})\colon S_\alpha(P_{n+2}) \underset{+1}{\xrightarrow{\phi}} H_0S_\alpha(Q_*) \underset{-1}{\subset} S_\alpha(Y_{n+1}).$$

Moreover, since

$$(\psi_{n+1}\bar{f}^X_{n+2} - \bar{f}^Y_{n+2}\varphi_{n+2})d^X_{n+3} = \psi_{n+1}\bar{f}^X_{n+2}d^X_{n+3} - \bar{f}^Y_{n+2}d^Y_{n+3}\varphi_{n+3}$$

we deduce that

$$\phi S_\alpha(d^X_{n+3}) = H_0S_\alpha(\varphi_*)\tilde{\kappa}^X - \tilde{\kappa}^Y S_\alpha(\varphi_{n+3}),$$

hence we are done. □

A consequence of Proposition 6.37 is that the construction of $\kappa(X_{\leq n}, P_*)$ in Definition 6.36 is independent of choices.

Proposition 6.38. *For an n-truncated Postnikov system $(X_{\leq n}, P_*)$, $\kappa(X_{\leq n}, P_*) = 0$ if an only if there exists an $(n+1)$-truncated Postnikov system $(X_{\leq n+1}, P_*)$ whose n-truncation is $(X_{\leq n}, P_*)$.*

Proof. If $(X_{\leq n+1}, P_*)$ exists we can take $\bar{f}_{n+2} = f_{n+2}$, hence the cocycle condition $f_{n+2}d_{n+3} = 0$ implies that $\tilde{\kappa} = 0$, so $\kappa(X_{\leq n}, P_*) = 0$.

Assume now that $\kappa(X_{\leq n}, P_*) = 0$. Suppose that we have made the necessary choices for the construction of $\tilde{\kappa}$. Since $\kappa(X_{\leq n}, P_*) = 0$ there exists a degree $+1$ morphism $\zeta\colon S_\alpha(P_{n+2}) \to H_0S_\alpha(P_*)$ such that $\tilde{\kappa} = \zeta d_{n+3}$. The composite

$$S_\alpha(P_{n+2}) \underset{+1}{\xrightarrow{\zeta}} H_0S_\alpha(P_*) \underset{-1}{\subset} S_\alpha(X_{n+1})$$

is the image by S_α of a unique $\phi\colon P_{n+2} \to X_{n+1}$. The equation $\tilde{\kappa} = \zeta d_{n+3}$ translates into $\phi d_{n+3} = \bar{f}_{n+2}d_{n+3}$. Hence i_{n+1}, q_{n+1} and $f_{n+2} = \bar{f}_{n+2} - \phi$ extend $(X_{\leq n}, P_*)$ to an $(n+1)$-truncated Postnikov system. □

Definition 6.39. Consider a couple of n-truncated Postnikov systems $(X_{\leq n}, P_*)$ and $(Y_{\leq n}, Q_*)$, $n > 0$, and a morphism between their $(n-1)$-truncations,

$$(\psi_{\leq n-1}, \varphi_*)\colon (X_{\leq n-1}, P_*) \longrightarrow (Y_{\leq n-1}, Q_*).$$

Take ψ'_n extending ψ_{n-1} and φ_n to an exact triangle morphism

$$\begin{tikzcd}
\cdots & X_{n-1} \arrow[rr] \arrow[dd, "\psi_{n-1}"'] & & X_n \arrow[dl] \arrow[dd, dashed, "\psi'_n"] & & & \\
& & P_n \arrow[ul] \arrow[dd, "\varphi_n"] & & P_{n+1} \arrow[ul] \arrow[dd, "\varphi_{n+1}"] & P_{n+2} \arrow[l] \arrow[dd, "\varphi_{n+2}"] & \cdots \arrow[l] \\
\cdots & Y_{n-1} \arrow[rr] & & Y_n \arrow[dl] & & & \\
& & Q_n \arrow[ul] & & Q_{n+1} \arrow[ul] & Q_{n+2} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

The square containing ψ'_n and φ_{n+1} need not commute, however

$$q^Y_n\psi'_n f^X_{n+1} = \varphi_n q^X_n f^X_{n+1} = \varphi_n d^X_{n+1} = d^Y_{n+1}\varphi_{n+1} = q^Y_n f^Y_{n+1}\varphi_{n+1}.$$

Hence, by Lemma 6.34, $S_\alpha(\psi'_n f^X_{n+1} - f^Y_{n+1}\varphi_{n+1})$ factors through $\operatorname{Ker} S_\alpha(q^Y_n) = \operatorname{Im} S_\alpha(i^Y_n) \cong H_0 S_\alpha(Q_*)$,

$$S_\alpha(\psi'_n f^X_{n+1} - f^Y_{n+1}\varphi_{n+1}) \colon S_\alpha(P_{n+1}) \underset{+1}{\xrightarrow{\tilde\theta}} H_0 S_\alpha(Q_*) \underset{-1}{\subset} S_\alpha(Y_n).$$

The following equations show that $\tilde\theta S_\alpha(d^X_{n+2}) = 0$,

$$\begin{aligned}(\psi'_n f^X_{n+1} - f^Y_{n+1}\varphi_{n+1}) d^X_{n+2} &= \psi'_n f^X_{n+1} d^X_{n+2} - f^Y_{n+1}\varphi_{n+1} d^X_{n+2}\\ &= -f^Y_{n+1} d^Y_{n+2}\varphi_{n+2} = 0.\end{aligned}$$

Here we use the cocycle condition for both n-truncated Postnikov systems.

The *obstruction of the morphism* $(\psi_{\leq n-1}, \varphi_*)$ *relative to the initial* n*-truncated Postnikov systems* is the element

$$\theta_{(X_{\leq n}, P_*),(Y_{\leq n}, Q_*)}(\psi_{\leq n-1}, \varphi_*) \in \operatorname{Ext}^{n+1,-n}_{\alpha,\mathscr{C}}(H_0 S_\alpha(P_*), H_0 S_\alpha(Q_*))$$

represented by a morphism $\tilde\theta$ constructed as above. We often omit the subscript of θ so as not to overload the notation. Notice that this obstruction is additive in the morphism,

$$\theta(\psi_{\leq n-1} + \bar\psi_{\leq n-1}, \varphi_* + \bar\varphi_*) = \theta(\psi_{\leq n-1}, \varphi_*) + \theta(\bar\psi_{\leq n-1}, \bar\varphi_*).$$

The following lemma allows to speak of the obstruction of a homotopy class.

Lemma 6.40. *Given two* n*-truncated Postnikov systems* $(X_{\leq n}, P_*)$ *and* $(Y_{\leq n}, Q_*)$*,* $n > 0$*, and two homotopic morphisms between their* $(n-1)$*-truncations*

$$(\psi_{\leq n-1}, \varphi_*) \simeq (\bar\psi_{\leq n-1}, \bar\varphi_*) \colon (X_{\leq n-1}, P_*) \longrightarrow (Y_{\leq n-1}, Q_*),$$

their obstructions coincide $\theta(\psi_{\leq n-1}, \varphi_*) = \theta(\bar\psi_{\leq n-1}, \bar\varphi_*)$.

Proof. It is enough to check that the obstruction of a nullhomotopic morphism $(\psi_{\leq n-1}, \varphi_*) \simeq 0$ vanishes. Since it is nullhomotopic $0 = i^Y_n \psi_{n-1} = \psi'_n i^X_n$, so we can factor $\psi'_n = \phi q^X_n$. Moreover, φ_* is nullhomotopic, so $\varphi_{n+1} = h_{n+1} d^X_{n+1} + d^Y_{n+2} h_{n+2}$ for certain h_{n+1} and h_{n+2},

$$\begin{tikzcd}
\cdots & X_{n-1} \arrow[rr] \arrow[dd, "\psi_{n-1}"'] & & X_n \arrow[dl] \arrow[dd, dashed, "\psi'_n"] & & & \\
& & P_n \arrow[ul] \arrow[dr, dashed, "\phi"] \arrow[dd, "\varphi_n"'] \arrow[ddrr, dashed, "h_{n+1}"'] & & P_{n+1} \arrow[ul] \arrow[dd, "\varphi_{n+1}"'] \arrow[dr, dashed, "h_{n+2}"] & P_{n+2} \arrow[l] \arrow[dd, "\varphi_{n+2}"] & \cdots \arrow[l] \\
\cdots & Y_{n-1} \arrow[rr] & & Y_n \arrow[dl] & & & \\
& & Q_n \arrow[ul] & & Q_{n+1} \arrow[ul] & Q_{n+2} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Using the direct sum decomposition in Lemma 6.34 we obtain

$$\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = S_\alpha(\phi - f^Y_{n+1} h_{n+1}) \colon S_\alpha(P_n) \longrightarrow S_\alpha(Y_n) \cong H_0 S_\alpha(Q_*) \oplus \operatorname{Im} S_\alpha(f^Y_{n+1}).$$

Then $\tilde\theta = \xi_1 S_\alpha(d^X_{n+1})$ since

$$\begin{aligned}(\phi - f^Y_{n+1} h_{n+1}) d^X_{n+1} &= \phi d^X_{n+1} - f^Y_{n+1} h_{n+1} d^X_{n+1} - f^Y_{n+1} d^Y_{n+2} h_{n+2}\\ &= \phi q^X_n f^X_{n+1} - f^Y_{n+1}(h_{n+1} d^X_{n+1} + d^Y_{n+2} h_{n+2})\\ &= \psi'_n f^X_{n+1} - f^Y_{n+1}\varphi_{n+1}.\end{aligned}$$

Here we use the cocycle condition $f^Y_{n+1} d^Y_{n+2} = 0$. Therefore $\theta(\psi_{\leq n-1}, \varphi_*) = 0$. $\square$

As consequence of Lemma 6.40, the obstruction of a morphism does not depend on choices.

Proposition 6.41. *With the notation in Definition 6.39,*

$$\theta_{(X_{\leq n},P_*),(Y_{\leq n},Q_*)}(\psi_{\leq n-1},\varphi_*)=0$$

if an only if there exists a morphism $\psi_n\colon X_n\to Y_n$ extending $(\psi_{\leq n-1},\varphi_*)$ to a morphism $(\psi_{\leq n},\varphi_*)\colon (X_{\leq n},P_*)\to(Y_{\leq n},Q_*)$ of n-truncated Postnikov systems.

Proof. If $(\psi_{\leq n},\varphi_*)$ extends the given morphism we can take $\psi'_n=\psi_n$, hence $\tilde\theta=\psi_n f^X_{n+1}-f^Y_{n+1}\varphi_{n+1}=0$ and the obstruction vanishes.

Conversely, if the obstruction vanishes take $\xi\colon S_\alpha(P_n)\to H_0S_\alpha(Q_*)$ with $\tilde\theta=\xi S_\alpha(d^X_{n+1})$. The composite

$$S_\alpha(P_{n+1})\xrightarrow{\xi}H_0S_\alpha(Q_*)\underset{-1}{\subset}S_\alpha(Y_n)$$

is the image by S_α of a unique $\phi\colon P_n\overset{-1}{\to}Y_n$, which must satisfy the two following equations

$$q^Y_n\phi=0,\qquad \phi d^X_{n+1}=\psi'_nf^X_{n+1}-f^Y_{n+1}\varphi_{n+1}.$$

We can take $\psi_n=\psi'_n-\phi q^X_n$, since

$$\begin{aligned}
\psi_n i^X_n&=(\psi'_n-\phi q^X_n)i^X_n=\psi'_ni^X_n=i^Y_n\psi_{n-1},\\
q^Y_n\psi_n&=q^Y_n(\psi'_n-\phi d^X_n)=q^Y_n\psi'_n=\varphi_nq^X_n,\\
\psi_nf^X_{n+1}&=(\psi'_n-\phi q^X_n)f^X_{n+1}=\psi'_nf^X_{n+1}-\phi d^X_{n+1}\\
&=\psi'_nf^X_{n+1}-(\psi'_nf^X_{n+1}-f^Y_{n+1}\varphi_{n+1})=f^Y_{n+1}\varphi_{n+1}.
\end{aligned}$$

□

The following result shows that the obstruction θ in Definition 6.39 is a derivation.

Proposition 6.42. *Given three n-truncated Postnikov systems $(X_{\leq n},P_*)$, $(Y_{\leq n},Q_*)$, and $(Z_{\leq n},R_*)$, and two composable morphisms between their $(n-1)$-truncations,*

$$(X_{\leq n-1},P_*)\xrightarrow{(\psi_{\leq n-1},\varphi_*)}(Y_{\leq n-1},Q_*)\xrightarrow{(\bar\psi_{\leq n-1},\bar\varphi_*)}(Z_{\leq n-1},R_*),$$

the following equation holds in $\operatorname{Ext}^{n+1,-n}_{\alpha,\mathscr{C}}(H_0S_\alpha(P_*),H_0S_\alpha(R_*))$,

$$\theta((\bar\psi_{\leq n-1},\bar\varphi_*)(\psi_{\leq n-1},\varphi_*))=\theta(\bar\psi_{\leq n-1},\bar\varphi_*)\cdot H_0S_\alpha(\varphi_*)+H_0S_\alpha(\bar\varphi_*)\cdot\theta(\psi_{\leq n-1},\varphi_*).$$

Proof. Assume we have chosen ψ'_n and $\bar{\psi}'_n$ to define the morphisms $\tilde{\theta}^{\psi}$ and $\tilde{\theta}^{\bar{\psi}}$ representing the obstructions of the two given morphisms,

$$\begin{tikzcd}
\cdots & X_{n-1} \arrow[rr] \arrow[dd, "\psi_{n-1}"'] & & X_n \arrow[dl] \arrow[dd, dashed, "\psi'_n"] & & & \\
& & P_n \arrow[ul] \arrow[dd, "\varphi_n"] & & P_{n+1} \arrow[ul] \arrow[dd, "\varphi_{n+1}"] & P_{n+2} \arrow[l] \arrow[dd, "\varphi_{n+2}"] & \cdots \arrow[l] \\
\cdots & Y_{n-1} \arrow[rr] \arrow[dd, "\bar{\psi}_{n-1}"'] & & Y_n \arrow[dl] \arrow[dd, dashed, "\bar{\psi}'_n"] & & & \\
& & Q_n \arrow[ul] \arrow[dd, "\bar{\varphi}_n"] & & Q_{n+1} \arrow[ul] \arrow[dd, "\bar{\varphi}_{n+1}"] & Q_{n+2} \arrow[l] \arrow[dd, "\bar{\varphi}_{n+2}"] & \cdots \arrow[l] \\
\cdots & Z_{n-1} \arrow[rr] & & Z_n \arrow[dl] & & & \\
& & R_n \arrow[ul] & & R_{n+1} \arrow[ul] & R_{n+2} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

We can take $\bar{\psi}'_{n+1}\psi'_{n+1}$ to define the morphism $\tilde{\theta}^{\bar{\psi}\psi}$ representing the obstruction of the composition. With this choice, the equation already holds for representatives,

$$\tilde{\theta}^{\bar{\psi}\psi} = \tilde{\theta}^{\bar{\psi}} S_\alpha(\varphi_{n+1}) + H_0 S_\alpha(\bar{\varphi}_*)\tilde{\theta}^{\psi},$$

since

$$\begin{aligned}
&(\bar{\psi}'_n f^Y_{n+1} - f^Z_{n+1}\bar{\varphi}_{n+1})\varphi_{n+1} + \bar{\psi}'_n(\psi'_n f^X_{n+1} - f^Y_{n+1}\varphi_{n+1}) \\
&\quad = \bar{\psi}'_n f^Y_{n+1}\varphi_{n+1} - f^Z_{n+1}\bar{\varphi}_{n+1}\varphi_{n+1} + \bar{\psi}'_n\psi'_n f^X_{n+1} - \bar{\psi}'_n f^Y_{n+1}\varphi_{n+1} \\
&\quad = (\bar{\psi}'_n\psi'_n) f^X_{n+1} - f^Z_{n+1}(\bar{\varphi}_{n+1}\varphi_{n+1}).
\end{aligned}$$

□

Proposition 6.43. *For any n-truncated Postnikov system $(X_{\leq n}, P_*)$ and any*

$$\zeta \in \operatorname{Ext}^{n+1,-n}_{\alpha,\mathscr{C}}(H_0 S_\alpha(P_*), H_0 S_\alpha(P_*))$$

there exists another n-truncated Postnikov system $(Y_{\leq n}, Q_*)$ with the same $(n-1)$-truncation $(X_{\leq n-1}, P_*) = (Y_{\leq n-1}, Q_*)$ such that

$$\theta_{(X_{\leq n},P_*),(Y_{\leq n},Q_*)}(\mathrm{id}_{(X_{\leq n-1},P_*)}) = \zeta.$$

Proof. We define the n-truncated Postnikov system $(Y_{\leq n}, Q_*)$ as follows, $X_k = Y_k$, $f^X_k = f^Y_k$, $i^X_k = i^Y_k$, $q^X_k = q^Y_k$, $0 \leq k \leq n$, $P_k = Q_k$, $k \geq 0$, $d^X_k = d^Y_k$, $k \geq n+2$. It is only left to define f^Y_{n+1}.

Choose a morphism $\tilde{\zeta}\colon S_\alpha(P_{n+1}) \overset{+1}{\to} H_0 S_\alpha(P_*)$ representing ζ. The composite

$$S_\alpha(P_{n+1}) \underset{+1}{\xrightarrow{\tilde{\zeta}}} H_0 S_\alpha(Q_*) \underset{-1}{\subset} S_\alpha(X_n)$$

is the image by S_α of a unique $\phi\colon P_{n+1} \to X_n$, which must satisfy $q^X_n\phi = 0$ and $\phi d_{n+2} = 0$, since $\tilde{\zeta} S_\alpha(d_{n+2}) = 0$. The morphism $f^Y_{n+1} = f^X_{n+1} - \phi$ yields an n-truncated Postnikov system $(Y_{\leq n}, Q_*)$ since the cocycle condition holds,

$$f^Y_{n+1} d_{n+2} = (f^X_{n+1} - \phi)d_{n+2} = f^X_{n+1}d_{n+2} - \phi d_{n+2} = 0 - 0 = 0.$$

To show that its $(n-1)$-truncation is $(X_{\leq n-1}, P_*)$, it is enough to notice that $d^Y_{n+1} = q^Y_n f^Y_{n+1} = q^X_n(f^X_{n+1} - \phi) = q^X_n f^X_{n+1} - 0 = d^X_{n+1}$. In order to compute the obstruction of $\mathrm{id}_{(X_{\leq n-1},P_*)}$ we can take $\psi'_n = \mathrm{id}_{X_n}$, so $\tilde{\theta} = \tilde{\zeta}$ and the obstruction is ζ. □

Definition 6.44. Given a pair of n-truncated Postnikov systems $(X_{\leq n}, P_*)$ and $(Y_{\leq n}, Q_*)$, $n > 0$, any degree 0 morphism $\tilde{\zeta}\colon S_\alpha(P_n) \to H_0 S_\alpha(Q_*)$ with $\tilde{\zeta} S_\alpha(d^X_{n+1}) = 0$ gives rise to a morphism

$$\bar{\imath}(\tilde{\zeta})\colon (X_{\leq n}, P_*) \longrightarrow (Y_{\leq n}, Q_*)$$

whose only non-trivial component is $g_{\tilde{\zeta}} q^X_n \colon X_n \to Y_n$,

$$\begin{tikzcd}
\cdots & X_{n-1} \arrow[rr] \arrow[dd, "0"'] & & X_n \arrow[dl] \arrow[dd, "g_{\tilde{\zeta}} q^X_n"] & & & \\
& & P_n \arrow[ul] \arrow[dd, "0"] \arrow[dr, dashed, "g_{\tilde{\zeta}}"] & & P_{n+1} \arrow[ul] \arrow[dd, "0"] & P_{n+2} \arrow[l] \arrow[dd, "0"] & \cdots \arrow[l] \\
\cdots & Y_{n-1} \arrow[rr] & & Y_n \arrow[dl] & & & \\
& & Q_n \arrow[ul] & & Q_{n+1} \arrow[ul] & Q_{n+2} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Here $g_{\tilde{\zeta}}\colon P_n \overset{-1}{\to} Y_n$ is the morphism whose image by S_α is

$$S_\alpha(P_n) \xrightarrow{\tilde{\zeta}} H_0 S_\alpha(Q_*) \overset{-1}{\subset} S_\alpha(Y_n).$$

This construction defines a natural homomorphism

$$\bar{\imath}\colon \operatorname{Ker} \operatorname{Hom}^0_{\alpha,\mathscr{C}}(S_\alpha(d^X_{n+1}), H_0 S_\alpha(Q_*)) \longrightarrow \mathbf{Post}_n((X_{\leq n}, P_*), (Y_{\leq n}, Q_*)).$$

Proposition 6.45. *The natural homomorphism $\bar{\imath}$ factors as*

$$\imath\colon \operatorname{Ext}^{n,-n}_{\alpha,\mathscr{C}}(H_0 S_\alpha(P_*), H_0 S_\alpha(Q_*)) \longrightarrow \mathbf{Post}_n((X_{\leq n}, P_*), (Y_{\leq n}, Q_*)).$$

Proof. It is enough to notice that if $\tilde{\zeta}$ factors through $S_\alpha(d^X_n)$ then $\bar{\imath}(\tilde{\zeta}) = 0$. This follows from $d^X_n q^X_n = q^X_{n-1} i^X_n q^X_n = q^X_{n-1} 0 = 0$. □

The kernel of $\imath = \imath_{(X_{\leq n}, P_*), (Y_{\leq n}, Q_*)}$ and of its composition with the natural projection onto the homotopy category $\mathbf{Post}^{\simeq}_n$ can be computed by means of spectral sequences associated to the Postnikov system $(X_{\leq n}, P_*)$. We omit the details to avoid further technicalities, compare [Bau89, page 340 and VI.5.16].

Proposition 6.46. *Given a morphism of n-truncated Postnikov systems*

$$(\psi_{\leq n}, \varphi_*)\colon (X_{\leq n}, P_*) \longrightarrow (Y_{\leq n}, Q_*),$$

its $(n-1)$-truncation $(\psi_{\leq n-1}, \varphi_*)$ is nullhomotopic if and only if $(\psi_{\leq n}, \varphi_*)$ is homotopic to a morphism in the image of $\imath$ in Proposition 6.45.

Proof. The truncation of a morphism in the image of $\imath$ is trivial. Conversely, if $(\psi_{\leq n-1}, \varphi_*)$ is nullhomotopic then $0 = i^Y_n \psi_{n-1} = \psi_n i^X_n$, so we can factor $\psi_n = \phi q^X_n$. Moreover, φ_* is nullhomotopic, so $\varphi_{n+1} = h_{n+1} d^X_{n+1} + d^Y_{n+2} h_{n+2}$ for certain h_{n+1} and h_{n+2},

$$\begin{tikzcd}
\cdots & X_{n-1} \arrow[rr] \arrow[dd, "\psi_{n-1}"'] & & X_n \arrow[dl] \arrow[dd, "\psi_n"] & & & \\
& & P_n \arrow[ul] \arrow[dd, "\varphi_n"'] \arrow[dr, dashed, "\phi"] & & P_{n+1} \arrow[ul] \arrow[dd, "\varphi_{n+1}"'] \arrow[dr, dashed, "h_{n+2}"] & P_{n+2} \arrow[l] \arrow[dd, "\varphi_{n+2}"] & \cdots \arrow[l] \\
\cdots & Y_{n-1} \arrow[rr] & & Y_n \arrow[dl] \arrow[dr, dashed, "h_{n+1}"'] & & & \\
& & Q_n \arrow[ul] & & Q_{n+1} \arrow[ul] & Q_{n+2} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

If we denote $\gamma=\phi-f^Y_{n+1}h_{n+1}$ we have that $\gamma d^X_{n+1}=0$, since

$$\begin{aligned}\phi d^X_{n+1}&=\phi q^X_n f^X_{n+1}=\psi_n f^X_{n+1}=f^Y_{n+1}\varphi_{n+1}\\&=f^Y_{n+1}(h_{n+1}d^X_{n+1}+d^Y_{n+2}h_{n+2})=f^Y_{n+1}h_{n+1}d^X_{n+1}.\end{aligned}$$

Here we use the cocycle condition $f^Y_{n+1}d^Y_{n+2}=0$.

Using the direct sum decomposition in Lemma 6.34,

$$\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=S_\alpha(\gamma)\colon S_\alpha(P_n)\longrightarrow S_\alpha(Y_n)\cong H_0S_\alpha(Q_*)\oplus\operatorname{Im}S_\alpha(f^Y_{n+1}).$$

Since $\gamma d^X_{n+1}=0$ we have $\xi_k S_\alpha(d^X_{n+1})=0$, $k=1,2$. Let us check that $\bar\imath(\xi_1)$ is homotopic to $(\psi_{\leq n},\varphi_*)$. Notice that, since $(\psi_{\leq n-1},\varphi_*)$ is nullhomotopic, we only have to check that $\psi_n-g_{\xi_1}q^X_n=(\phi-g_{\xi_1})q^X_n$ factors through $f^Y_{n+1}\colon Q_{n+1}\to Y_n$ where g_{ξ_1} is the morphism whose image by S_α is ξ_1. This is obvious since by construction the image of $S_\alpha(\phi-g_{\xi_1})=S_\alpha(\phi)-\xi_1$ lies on $\operatorname{Im}S_\alpha(f^Y_{n+1})$ in the previous direct sum decomposition. □

6.7. **The obstruction of a module.** In this short section we analyze the most basic of the obstructions in Section 6.6.

Definition 6.47. The *obstruction of an α-continuous $\mathscr{C}$-module M* is the obstruction of a 0-truncated Postnikov system (X_0,P_*) with homology $H_0S_\alpha(P_*)=M$,

$$\kappa(M)=\kappa(X_0,P_*)\in\operatorname{Ext}^{3,-1}_{\alpha,\mathscr{C}}(M,M).$$

The following characterization of this obstruction extends [BKS04, Theorem 3.7].

Proposition 6.48. *Given an α-continuous $\mathscr{C}$-module M, $\kappa(M)=0$ if an only if M is a retract of a restricted representable functor $S_\alpha(X)$.*

Proof. If $\kappa(M)=0$ we can extend (X_0,P_*) to a 1-truncated Postnikov system $(X_{\leq 1},P_*)$ by Proposition 6.38, and Lemma 6.34 shows that M is a direct summand of $S_\alpha(X_1)$. Conversely, we always have $\kappa(S_\alpha(X))=0$ from the existence of Postnikov resolutions, see Corollary 6.11 and Proposition 6.38. Moreover, if

$$M\underset{r}{\overset{i}{\rightleftarrows}}S_\alpha(X)\qquad\qquad ri=\mathrm{id}_M$$

is a retraction, then, by Proposition 6.37,

$$\kappa(M)=\mathrm{id}_M\cdot\kappa(M)=ri\cdot\kappa(M)=r\cdot\kappa(S_\alpha(X))\cdot i=0.$$

□

Corollary 6.49. *If S_α is full, then an α-continuous $\mathscr{C}$-module M is isomorphic to a restricted representable functor, $M\cong S_\alpha(X)$, if and only if $\kappa(M)=0$.*

Proof. If M is a restricted representable functor, then $\kappa(M)=0$ by Proposition 6.48. Conversely, if $\kappa(M)=0$, then we have a retraction

$$M\underset{r}{\overset{i}{\rightleftarrows}}S_\alpha(X)\qquad\qquad ri=\mathrm{id}_M\,.$$

Since S_α is full $ir\colon S_\alpha(X)\to S_\alpha(X)$ is the image by S_α of some $f\colon X\to X$. One can check, as in the proof of Theorem 6.29, that M is isomorphic to the image by S_α of

$$\operatorname{Hocolim}(X\xrightarrow{f}X\xrightarrow{f}X\xrightarrow{f}\cdots).$$

□

6.8. **Connection with the Adams spectral sequence.** Given a pair of objects X and Y in $\mathscr{T}$, the *Adams spectral sequence* is a conditionally convergent cohomological spectral sequence abutting to $\mathscr{T}^*(X,Y)$ with E_2-term

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{\alpha,\mathscr{C}}(S_\alpha(X), S_\alpha(Y)),$$

cf. [Chr98, Section 4]. It is defined by the exact couple

$$\begin{tikzcd} \mathscr{T}(W_*,Y) \arrow[dr, "\mathscr{T}(g_*{,}Y)"', "(+1{,}0)"] & & \mathscr{T}(W_*,Y) \arrow[ll, "\mathscr{T}(j_*{,}Y)"', "(-1{,}0)"] \\ & \mathscr{T}(P_*,Y) \arrow[ur, "\mathscr{T}(r_*{,}Y)"', "(0{,}-1)"] & \end{tikzcd}$$

associated to an Adams resolution (X, W_*, P_*) of X. Here we set $W_{-1} = X$. The induced decreasing filtration of $\mathscr{T}^*(X,Y)$ is the filtration by powers of the ideal $\mathscr{I}$ of phantom maps.

The following result relates the obstruction to the lifting of morphisms along t_{n-1} in Definition 6.39 with the differentials of the Adams spectral sequence.

Proposition 6.50. *Given Postnikov resolutions (X, X_*, P_*) and (Y, Y_*, Q_*) and a morphism of $(n-1)$-truncated Postnikov systems $(\psi_{\leq n-1}, \varphi_*)\colon (X_{\leq n-1}, P_*) \to (Y_{\leq n-1}, Q_*)$, $n > 0$, the differentials of the previous Adams spectral sequence satisfy*

$$d_k(H_0S_\alpha(\varphi_*)\colon S_\alpha(X) \longrightarrow S_\alpha(Y)) = \begin{cases} 0 & \text{if } 2 \leq k \leq n \\ \theta_{(X_{\leq n}, P_*),(Y_{\leq n}, Q_*)}(\psi_{\leq n-1}, \varphi_*) & \text{if } k = n+1. \end{cases}$$

In particular, if $f\colon S_\alpha(X) \to S_\alpha(Y)$ is a morphism satisfying $d_k(f) = 0$ for $1 < k \leq n$, then there exists a morphism of $(n-1)$-truncated Postnikov systems as above with $f = H_0S_\alpha(\varphi_*)$.

Proof. The second part of the statement follows from the first part and Proposition 6.41. Let us deal with the first part.

Take an Adams resolution (X, W_*, P_*) adapted to (X, X_*, P_*) in the sense of Lemma 6.10. The morphisms ϕ_n and id_{P_n}, $n \geq 0$, define a morphism between the previous exact couple and the exact couple

$$\begin{tikzcd} \mathscr{T}(X_*,Y) \arrow[dr, "\mathscr{T}(f_*{,}Y)"', "(+1{,}0)"] & & \mathscr{T}(X_*,Y) \arrow[ll, "\mathscr{T}(i_*{,}Y)"', "(-1{,}0)"] \\ & \mathscr{T}(P_*,Y) \arrow[ur, "\mathscr{T}(q_*{,}Y)"', "(0{,}-1)"] & \end{tikzcd}$$

associated to the Postnikov system (X_*, P_*). This morphism is the identity on E^1-terms, and hence on E^k-terms for all $k \geq 1$. We can therefore compute the differentials of the Adams spectral sequence by using this second exact couple.

Let $n = 1$. Since $H_0 S_\alpha(\varphi_*)$ is represented by $p_0^Y (q_0^Y)^{-1} \varphi_0 q_0^X$, using the second exact couple it is clear that $d_2(H_0 S_\alpha(\varphi_*))$ is represented by $p_1^Y \psi_1' f_2^X$.

$$\begin{tikzcd}
X & & & & & & & \\
& 0 \arrow[ul] \arrow[rr, "i_0^X"] \arrow[dddd, equal] & & X_0 \arrow[lll, bend right, "p_0^X"'] \arrow[rr, "i_1^X"] \arrow[dl, "+1", "q_0^X"'] \arrow[ddd, "\psi_0"] & & X_1 \arrow[lllll, bend right, "p_1^X"'] \arrow[dl, "+1", "q_1^X"'] \arrow[ddd, dashed, "\psi_1'"] & & \cdots \\
& & P_0 \arrow[ul] \arrow[ddd, "\varphi_0"] & & P_1 \arrow[ul] \arrow[ddd, "\varphi_1"] & & P_2 \arrow[ul, "f_2^X"] \arrow[ddd, "\varphi_2"] & \\
Y & & & & & & & \\
& 0 \arrow[ul] \arrow[rr, "i_0^Y"] & & Y_0 \arrow[lll, bend right, "p_0^Y"'] \arrow[rr, "i_1^Y"] \arrow[dl, "+1", "q_0^Y"'] & & Y_1 \arrow[lllll, bend right, "p_1^Y"'] \arrow[dl, "+1", "q_1^Y"'] & & \cdots \\
& & Q_0 \arrow[ul, "f_0^Y"] & & Q_1 \arrow[ul, "f_1^Y"] & & Q_2 \arrow[ul, "f_2^Y"] &
\end{tikzcd}$$

Moreover, $S_\alpha(p_1^Y \psi_1' f_2^X) = \tilde{\theta}$ by Lemma 6.15 since

$$p_1^Y(\psi_1' f_2^X - f_2^Y \varphi_2) = p_1^Y \psi_1' f_2^X - p_1^Y f_2^Y \varphi_2 = p_1^Y \psi_1' f_2^X - 0\varphi_2 = p_1^Y \psi_1' f_2^X.$$

If $n > 1$ then $\psi_1' = \psi_1$ and $p_1^Y \psi_1' f_2^X = p_1^Y f_2^Y \varphi_2 = 0\varphi_2 = 0$ by Lemma 6.15. In this way, by induction $d_k(H_0 S_\alpha(\varphi_*)) = 0$ for $1 < k \leq n$ and $d_{n+1}(H_0 S_\alpha(\varphi_*))$ is represented by $p_n^Y \psi_n' f_{n+1}^X$. Moreover, $S_\alpha(p_n^Y \psi_n' f_{n+1}^X) = \tilde{\theta}$ by Lemma 6.15 since

$$\begin{aligned} p_n^Y(\psi_n' f_{n+1}^X - f_2^Y \varphi_{n+1}) &= p_n^Y \psi_n' f_{n+1}^X - p_n^Y f_{n+1}^Y \varphi_{n+1} \\ &= p_n^Y \psi_n' f_{n+1}^X - 0\varphi_{n+1} = p_n^Y \psi_n' f_{n+1}^X. \end{aligned}$$

□

7. The first obstruction of an extension of representables

A triangulated category is said to be *algebraic* if it is a full triangulated subcategory of the homotopy category $K(\mathscr{A})$ of some additive category $\mathscr{A}$, cf. [Kra07, §7.5].

Theorem 7.1. *Let $\mathscr{T}$ be an algebraic triangulated category. Suppose F is a $\mathscr{C}$-module fitting into a short exact sequence*

$$S_\alpha(Y) \overset{a}{\hookrightarrow} F \overset{b}{\twoheadrightarrow} S_\alpha(X)$$

classified by

$$e_F \in \operatorname{Ext}^{1,0}_{\mathscr{C}}(S_\alpha(X), S_\alpha(Y)).$$

Then the obstruction of F is

$$\kappa(F) = a \cdot d_2(e_F) \cdot b \in \operatorname{Ext}^{3,-1}_{\mathscr{C}}(F, F),$$

where d_2 is the second differential of the Adams spectral sequence in Section 6.8 abutting to $\mathscr{T}^*(X, Y)$.

This result is a paradigmatic example of a statement which makes sense for any triangulated category but which requires the use of models in its proof. The proof uses maps and homotopies in the category of complexes in $\mathscr{A}$, actually homotopy classes of homotopies suffice, but we will not get into such technicalities. Nevertheless, this suggests that it should be enough to assume that $\mathscr{T}$ is the homotopy category of a triangulated track category [BM08, BM09]. This includes topological triangulated categories, i.e. full triangulated subcategories of stable model categories. The proof in the non-additive setting is however more complicated. This is why we restrict to algebraic triangulated categories here. The proof is at the end of this section.

Definition 7.2. Let $C(\mathscr{A})$ be the category of chain complexes in an additive category $\mathscr{A}$. Differentials of chain complexes in $\mathscr{A}$ are denoted by ∂ and have degree -1. We add a superscript ∂^A if we need to specify the complex A. A *type* 0 *standard exact triangle starting at* A is a diagram in $C(\mathscr{A})$

$$\begin{tikzcd} A \arrow[rr, "f"] & {} \arrow[d, phantom, "+1"] & B \arrow[dl, "i"] \\ & C_f \arrow[ul, "q"] & \end{tikzcd}$$

such that C_f is the mapping cone of f,

$$(C_f)_n = A_{n-1} \oplus B_n, \qquad \partial_n^{C_f} = \begin{pmatrix} -\partial^A_{n-1} & 0 \\ f_{n-1} & \partial^B_n \end{pmatrix},$$

and i and q are given by

$$B_n \xrightarrow{i_n = \binom{0}{1}} (C_f)_n = A_{n-1} \oplus B_n \xrightarrow{q_n = (1,0)} A_{n-1}.$$

The *type* 1 *standard exact triangle starting at* ΣA and the *type* 2 *standard exact triangle starting at* A are

$$\begin{tikzcd} \Sigma A \arrow[rr, "f", "+1"'] & & B \arrow[dl, "i"] \\ & C_f \arrow[ul, "q"] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "f"] & {} \arrow[d, phantom, "+1"] & B \arrow[dl, "i"] \\ & \Sigma^{-1} C_f \arrow[ul, "q"] & \end{tikzcd}$$

respectively. Notice that, in all cases, $qi = 0$ in $C(\mathscr{A})$.

Remark 7.3. Recall that a chain map $\binom{g}{h}\colon D \to C_f$ is the same as a chain map $g\colon D \to \Sigma A$, given by morphisms $g_n\colon D_n \to A_{n-1}$ with $\partial^A_{n-1} g_n + g_{n-1}\partial^D_n = 0$, together with a nullhomotopy $h\colon (\Sigma f)g \Rightarrow 0$, i.e. a sequence of morphisms $h_n\colon D_n \to B_n$ with $f_{n-1}g_n + \partial^B_n h_n = h_{n-1}\partial^D_n$. Similarly, a chain map $(h, g)\colon C_f \to D$ is simply a map $g\colon B \to D$ together with a nullhomotopy $h\colon gf \Rightarrow 0$.

Suppose for the rest of this section that $\mathscr{T}$ is algebraic, and fix an embedding $\mathscr{T} \subset K(\mathscr{A})$ which allows us to work with complexes in $\mathscr{A}$. The following lemma shows how to compute $\kappa(F)$ by means of chain homotopies.

Lemma 7.4. *Let F be an α-continuous $\mathscr{C}$-module and*

$$\cdots \to R_m \xrightarrow{d_m} R_{m-1} \to \cdots \to R_0$$

a sequence of morphisms in $C(\mathscr{A})$ whose homotopy classes lie in $\mathscr{T}$ and map by S_α to a resolution of F in $\mathrm{Mod}_\alpha(\mathscr{C})$. Let $h_m\colon d_m d_{m+1} \Rightarrow 0$ be nullhomotopies, $m = 1, 2,$

$$\begin{tikzcd}[column sep=large] R_3 \arrow[r, "d_3", "+1"'] \arrow[rr, bend right=40, "+2", "0"'] & R_2 \arrow[r, "d_2", "+1"'] \arrow[rr, bend left=40, "0", "+2"'] \arrow[d, phantom, "\Downarrow h_2"] & R_1 \arrow[r, "d_1", "+1"'] \arrow[u, phantom, "h_1 \Uparrow"] & R_0 \end{tikzcd}$$

The degree $+2$ chain morphism $R_3 \to R_0$ defined by the morphisms

$$h_{1,n-1}d_{3,n} - d_{1,n-1}h_{2,n} \colon R_{3,n} \longrightarrow R_{0,n-2}, \quad n \in \mathbb{Z},$$

represents $\kappa(F)$.

Proof. The nullhomotopies consist of morphisms $h_{m,n} \colon R_{m+1,n} \to R_{m-1,n-1}$ in $\mathscr{A}$, $n \in \mathbb{Z}$, with

$$d_{m,n-1}d_{m+1,n} = \partial_{n-1}^{R_{m-1}} h_{m,n} + h_{m,n-1}\partial_n^{R_{m+1}}.$$

Take $Z_0 = \Sigma R_0$, $f_1 = d_1$, and extend this morphism to a standard exact triangle of type 0 starting at R_1. For the definition of $\kappa(F)$ we can take $f_2' = \binom{d_2}{h_1} \colon R_2 \to Z_1 = C_{d_1}$ as in the following diagram

$$\begin{tikzcd}
\Sigma R_0 \arrow[rr, dashed, "i_1"] & & Z_1 \arrow[dl, dashed, "+1"', "q_1"] & & & & \\
& R_1 \arrow[ul, "d_1"] & & R_2 \arrow[ll, "+1", "d_2"'] \arrow[ul, dashed, "f_2'"'] & & R_3 \arrow[ll, "d_3", "+1"'] & \cdots \arrow[l]
\end{tikzcd}$$

Then $f_2'd_3$ is given by the following morphisms in $\mathscr{A}$, $n \in \mathbb{Z}$:

$$\begin{pmatrix} d_{2,n-1} \\ h_{1,n-1} \end{pmatrix} d_{3,n} = \begin{pmatrix} d_{2,n-1}d_{3,n} \\ h_{1,n-1}d_{3,n} \end{pmatrix} = \begin{pmatrix} \partial_{n-1}^{R_1} h_{2,n} + h_{2,n-1}\partial_n^{R_3} \\ h_{1,n-1}d_{3,n} \end{pmatrix}.$$

We can deform this representative of the composite $f_2'd_3$ in $\mathscr{T}$ by using the morphisms $\binom{h_{2,n}}{0} \colon R_{3,n} \to R_{1,n-1} \oplus R_{0,n}$, $n \in \mathbb{Z}$, obtaining a chain morphism in the same homotopy class defined by the following morphisms in $\mathscr{A}$, $n \in \mathbb{Z}$:

$$\begin{aligned}
&\begin{pmatrix} d_{2,n-1} \\ h_{2,n-1} \end{pmatrix} d_{3,n} - \begin{pmatrix} h_{2,n-1} \\ 0 \end{pmatrix} \partial_n^{R_3} - \partial_n^{C_{d_1}} \begin{pmatrix} h_{2,n} \\ 0 \end{pmatrix} \\
=&\begin{pmatrix} \partial_{n-1}^{R_1} h_{2,n} + h_{2,n-1}\partial_n^{R_3} \\ h_{1,n-1}d_{3,n} \end{pmatrix} - \begin{pmatrix} h_{2,n-1} \\ 0 \end{pmatrix} \partial_n^{R_3} - \begin{pmatrix} -\partial_{n-1}^{R_1} & 0 \\ d_{1,n-1} & \partial_n^{R_0} \end{pmatrix} \begin{pmatrix} h_{2,n} \\ 0 \end{pmatrix} \\
=&\begin{pmatrix} 0 \\ h_{1,n-1}d_{3,n} - d_{1,n-1}h_{2,n} \end{pmatrix},
\end{aligned}$$

hence we are done. $\square$

Lemma 7.5. *Let (X, X_*, P_*) be a Postnikov resolution whose underlying Postnikov system consists of type 2 standard triangles starting at X_{m-1}, $m \geq 0$, where $X_{-1} = 0$,*

$$\begin{tikzcd}
0 \arrow[rr, "i_0"] & & X_0 \arrow[rr, "i_1"] \arrow[dl, "q_0"', "+1"] & & X_1 \arrow[rr, "i_2"] \arrow[dl, "q_1"', "+1"] & & X_2 \arrow[rr, "i_3"] \arrow[dl, "q_2"', "+1"] & & X_3 \arrow[dl, "q_3"', "+1"] & \cdots \\
& P_0 \arrow[ul, "f_0"] & & P_1 \arrow[ul, "f_1"] & & P_2 \arrow[ul, "f_2"] & & P_3 \arrow[ul, "f_3"] & &
\end{tikzcd}$$

Given $e \in \operatorname{Ext}_{\alpha,\mathscr{C}}^{s,t}(S_\alpha(X), S_\alpha(Y)) = E_2^{s,t}$ represented by a chain map $\tilde{e} \colon P_s \to Y$ of degree $s+t$, if $l \colon \tilde{e}d_{s+1}^X \Rightarrow 0$ is a nullhomotopy, then the image of e by the Adams spectral sequence's second differential $d_2(e) \in \operatorname{Ext}_{\alpha,\mathscr{C}}^{s+2,t-1}(S_\alpha(X), S_\alpha(Y)) = E_2^{s+2,t-1}$ is represented by the chain map $P_{s+2} \to Y$ of degree $s+t+1$ defined by the following morphisms, $n \in \mathbb{Z}$:

$$-l_{n-1}d_{s+2,n}^X \colon P_{s+2,n} \longrightarrow Y_{n-s-t-1}.$$

Proof. The nullhomotopy l is given by morphisms $l_n\colon P_{s+1,n} \to Y_{n-s-t}$ in $\mathscr{A}$ satisfying $\tilde{e}_{n-1}d^X_{s+1,n} = \partial^Y_{n-s-t}l_n + l_{n-1}\partial^{P_{s+1}}_n$, $n \in \mathbb{Z}$. This nullhomotopy and $\tilde{e}q^X_s$ define a degree $s+t+1$ morphism $(l, \tilde{e}q^X_s)\colon C_{f^X_{s+1}} \to Y$. Since the extact triangles of (X_*, P_*) are standard of type 2, P_{s+1} is the the desuspension of the mapping cone of i^X_{s+1}, and $C_{f^X_{s+1}}$ is given by

$$(C_{f^X_{s+1}})_n = X_{s,n-1} \oplus X_{s+1,n} \oplus X_{s,n}, \qquad \partial_n^{C_{f^X_{s+1}}} = \left(\begin{array}{cc|c} -\partial^{X_s}_{n-1} & 0 & 0 \\ i_{s+1,n} & \partial^{X_{s+1}}_n & 0 \\ \hline 1 & 0 & \partial^{X_s}_n \end{array}\right).$$

The inclusion of the middle direct summands

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : X_{s+1,n} \longrightarrow (C_{f^X_{s+1}})_n = X_{s,n-1} \oplus X_{s+1,n} \oplus X_{s,n}$$

yield a homotopy equivalence $X_{s+1} \xrightarrow{\sim} C_{f^X_{s+1}}$ such that the triangle

$$\begin{tikzcd} X_s \arrow[r, "i_{s+1}"] \arrow[dr, "\text{inclusion into mapping cone}"'] & X_{s+1} \arrow[d, "\sim"] \\ & C_{f^X_{s+1}} \end{tikzcd}$$

anticommutes up to the homotopy given by the morphisms

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : X_{s,n} \longrightarrow (C_{f^X_{s+1}})_{n+1} = X_{s,n} \oplus X_{s+1,n+1} \oplus X_{s,n+1}.$$

Hence $-d_2(e)$ is represented by

$$P_{s+2} \xrightarrow{f^X_{s+2}} X_{s+1} \xrightarrow{\sim} C_{f^X_{s+1}} \xrightarrow[p+s+1]{(l,\tilde{e}q^X_s)} Y.$$

This composite is defined by the morphisms $l_{n-1}d^X_{s+2,n}$, $n \in \mathbb{Z}$, hence we are done. □

Remark 7.6. It is always possible to represent a Postnikov system by type 2 standard triangles as in the statement of Lemma 7.5. Moreover, in the conditions of that statement, if $\tilde{e}\colon P_s \to Y$ represents an element in $\operatorname{Ext}^{s,t}_{\alpha,\mathscr{C}}(S_\alpha(X), S_\alpha(Y))$ there must exist a nullhomotopy $l\colon \tilde{e}d^X_{s+1} \Rightarrow 0$ since $\tilde{e}d^X_{s+1} = 0$ in $\mathscr{T}$.

Proof of Theorem 7.1. Take a Postnikov resolution (X, X_*, P_*) whose underlying Postnikov system (X_*, P_*) consists of type 2 standard triangles starting at X_{m-1}, $m \geq 0$, and an Adams resolution (Y, W_*, Q_*) consisting of type 2 standard triangles starting at Y and W_m, $m \geq 0$. By elementary homological algebra, there are degree $+1$ chain maps $s_m\colon P_m \to Q_{m-1}$, $m > 0$, such that the morphisms

$$d^Z_m = \begin{pmatrix} d^Y_m & s_m \\ 0 & d^X_m \end{pmatrix} : Q_m \oplus P_m \longrightarrow Q_{m-1} \oplus P_{m-1},$$

map to a projective resolution of F in $\operatorname{Mod}_\alpha(\mathscr{C})$. The element e_F is represented by $-S_\alpha(s_1)$. Since these matrices define differentials in $\mathscr{T}$, $d^X_m d^X_{m+1} = 0$, $d^Y_m d^Y_{m+1} = 0$, and $d^Y_m s_{m+1} + s_m d^X_{m+1} = 0$. The first two equations also hold at the level of chain maps by the properties of standard triangles. For the third equation, we choose an arbitrary nullhomotopy $k_m\colon d^Y_m s_{m+1} + s_m d^X_{m+1} \Rightarrow 0$, defined by morphisms $k_{m,n}\colon P_{m+1,n} \to Q_{m-1,n-1}$, $n \in \mathbb{Z}$, satisfying

$$d^Y_{m,n-1}s_{m+1,n} + s_{m,n-1}d^X_{m+1,n} = \partial^{Q_{m-1}}_{n-1}k_{m,n} + k_{m,n-1}\partial^{P_{m+1}}_n. \tag{7.7}$$

We take $h_m\colon d^Z_m d^Z_{m+1} \Rightarrow 0$, $m = 1, 2$, to be defined by

$$h_{m,n} = \begin{pmatrix} 0 & k_{m,n} \\ 0 & 0 \end{pmatrix}, \quad n \in \mathbb{Z}.$$

By Lemma 7.4 the following morphisms define a chain morphism representing $\kappa(F)$,

$$\begin{pmatrix} 0 & k_{1,n-1}d^X_{3,n} - d^Y_{1,n-1}k_{2,n} \\ 0 & 0 \end{pmatrix} : Q_{3,n} \oplus P_{3,n} \longrightarrow Q_{0,n-2} \oplus P_{0,n-2}.$$

This shows that if $x \in \mathrm{Ext}^{3,-1}_{\alpha,\mathscr{C}}(S_\alpha(X), S_\alpha(Y))$ is the element represented by the chain map defined by the following morphisms, $n \in \mathbb{Z}$,

$$g_{0,n-2}(k_{1,n-1}d^X_{3,n} - d^Y_{1,n-1}k_{2,n}) = g_{0,n-2}k_{1,n-1}d^X_{3,n} - 0k_{2,n} = g_{0,n-2}k_{1,n-1}d^X_{3,n},$$

then

$$\kappa(F) = a \cdot x \cdot b.$$

We now identify this x with $d_2(e_F)$.

Take

$$\tilde{e}\colon P_1 \underset{+1}{\xrightarrow{s_1}} Q_0 \xrightarrow{g_0} Y, \qquad l_n = g_{0,n-1}k_{1,n}, \quad n \in \mathbb{Z}.$$

We must check that l defined in this way is a homotopy. Indeed, since g_0 is a chain map

$$\begin{aligned}
\partial^Y_{n-1}l_n + l_{n-1}\partial^{P_2}_n &= \partial^Y_{n-1}g_{0,n-1}k_{1,n} + g_{0,n-2}k_{1,n-1}\partial^{P_2}_n \\
&= g_{0,n-2}\partial^{Q_0}_{n-1}k_{1,n} + g_{0,n-2}k_{1,n-1}\partial^{P_2}_n \\
&= g_{0,n-2}(\partial^{Q_0}_{n-1}k_{1,n} + k_{1,n-1}\partial^{P_2}_n) \\
&= g_{0,n-2}(d^Y_{1,n-1}s_{2,n} + s_{1,n-1}d^X_{2,n}) \text{ by (7.7)} \\
&= 0s_{2,n} + g_{0,n-2}s_{1,n-1}d^X_{2,n} \\
&= \tilde{e}_{n-1}d^X_{2,n}.
\end{aligned}$$

Hence $d_2(e_F) = x$ by Lemma 7.5. □

8. A characterization of α-compact objects

The following theorem is used in Sections 3 and 4 to prove that some categories satisfy $\mathrm{ARO}_{\aleph_1}$.

Theorem 8.1. *Let α be a regular cardinal. Suppose that β is a cardinal satisfying one of the following hypotheses:*

1. *$\beta = (\gamma^{<\delta})^+$ for some $\gamma \geq \operatorname{card} \mathscr{T}^\alpha$ and some regular cardinal $\delta \geq \alpha$.*
2. *$\beta > \operatorname{card} \mathscr{T}^\alpha$ is inaccessible.*

Then $\mathscr{T}^\beta$ is the full subcategory of objects Z such that $\mathscr{T}(Y, Z) < \beta$ for any Y in $\mathscr{T}^\alpha$.

One can easily produce cardinals satisfying (1), however the existence of cardinals as in (2) depends on large cardinal principles. Theorem 8.1 recovers Krause's [Kra02, Theorem C] by taking $\beta = (\gamma^{<\delta})^+$ as in (1) for $\gamma = \operatorname{card}\mathscr{T}^\alpha$ and $\delta = \alpha^+$, i.e. $\beta = ((\operatorname{card}\mathscr{T}^\alpha)^\alpha)^+$. Notice that the smallest cardinal β we can take as in (1) is $\beta = ((\operatorname{card}\mathscr{T}^\alpha)^{<\alpha})^+$, which is smaller than Krause's choice.

Lemma 8.2. *Let β be as in the statement of Theorem 8.1. Given a set I of* $\operatorname{card} I < \beta$ *and objects X_i, Y in $\mathscr{T}^\alpha$, $i \in I$, then* $\operatorname{card}\mathscr{T}(Y, \coprod_{i\in I} X_i) < \beta$.

Proof. Notice that $\beta > \operatorname{card}\mathscr{T}^\alpha$ in both cases. Since Y is α-small,

$$\mathscr{T}(Y, \coprod_{i\in I} X_i) = \operatorname*{colim}_{\substack{J\subset I\\ \operatorname{card} J<\alpha}} \mathscr{T}^\alpha(Y, \coprod_{i\in I} X_i).$$

The cardinal of this set is bounded above by $(\operatorname{card} I)^{<\alpha}\cdot \operatorname{card}\mathscr{T}^\alpha$, so it is enough to check that $(\operatorname{card} I)^{<\alpha} < \beta$. If β satisfies condition (2), the result follows from the strong limit property. Otherwise, $(\operatorname{card} I)^{<\alpha} \le (\gamma^{<\delta})^{<\alpha} = \gamma^{<\delta} < \beta$ by [AR94, Lemma 2.10]. □

The following lemma is obvious.

Lemma 8.3. *Let S be a class of objects in $\mathscr{T}$ closed under (de)suspensions, $\Sigma S = S$, and β an infinite cardinal. The full subcategory of objects Z such that $\mathscr{T}(Y, Z) < \beta$ for all $Y \in S$ is triangulated.*

We are now ready to prove Theorem 8.1.

Proof of Theorem 8.1. Denote $\mathscr{S}$ the full subcategory of $\mathscr{T}$ spanned by the objects Z such that $\mathscr{T}(Y, Z) < \beta$ for any Y in $\mathscr{T}^\alpha$. This subcategory is triangulated by Lemma 8.3. We claim that Z is in $\mathscr{S}$ if and only if there is an morphism $g_0\colon P_0 = \coprod_{i\in I} X_i \to Z$ with X_i in $\mathscr{T}^\alpha$ and $\operatorname{card} I < \beta$, such that $S_\alpha(g_0)$ is an epimorphism. If such a morphism exists, then for any Y in $\mathscr{T}^\alpha$, $\operatorname{card}\mathscr{T}(Y, Z) \le \operatorname{card}\mathscr{T}(Y, \coprod_{i\in I} X_i) < \beta$ by Lemma 8.2, so Z is in $\mathscr{S}$. Conversely, if Z is in $\mathscr{S}$, consider the evaluation morphism

$$g\colon P = \coprod_{\substack{Y \text{ in } \mathscr{T}^\alpha\\ \mathscr{T}(Y,Z)}} Y \longrightarrow Z.$$

The coproduct is indexed by a set of cardinality $\le \sum_{\operatorname{card}\mathscr{T}^\alpha} \operatorname{card}\mathscr{T}(Y, Z) < \beta$, since β is regular, and $S_\alpha(g)$ is clearly an epimorphism.

We now prove that $\mathscr{S} = \mathscr{T}^\alpha$. Given an object Z in $\mathscr{S}$, we can construct, as in Remark 6.7, an Adams resolution (Z, W_*, P_*) where each P_n is a direct sum of $< \beta$ objects in $\mathscr{T}^\alpha \subset \mathscr{T}^\beta$, so each P_n is in $\mathscr{T}^\beta$. Let (Z, Z_*, P_*) be an associated Postnikov resolution, as in Lemma 6.10. It can be seen by induction that each Z_n is in $\mathscr{T}^\beta$ since we have exact triangles $P_n \to Z_{n-1} \to Z_n \to \Sigma P_n$. Hence $Z = \operatorname{Hocolim}_n Z_n$ is also in $\mathscr{T}^\beta$ because, since $\beta > \aleph_0$, $\mathscr{T}^\beta$ has countable coproducts. This proves $\mathscr{S} \subset \mathscr{T}^\beta$.

Since $\mathscr{T}^\beta$ is the smallest β-localizing subcategory containing a set of α-compact generators, in order to show $\mathscr{T}^\beta \subset \mathscr{S}$ it is enough to see that $\mathscr{S}$ is β-localizing, i.e. closed under coproduct of $< \beta$ objects. Let $\{Z_j\}_{j\in J}$ be a set of objects in $\mathscr{S}$ with $\operatorname{card} J < \beta$. By the first part of the proof there are morphisms $g_j\colon P_j \to Z_j$ such that P_j is a coproduct of $< \beta$ objects in $\mathscr{T}^\alpha$ and $S_\alpha(g_j)$ is an epimorphism for all $i \in J$. Hence, the source of $\coprod_{j\in J} g_j\colon \coprod_{j\in J} P_j \to \coprod_{j\in J} Z_j$ is also a coproduct of $<\beta$ objects in $\mathscr{T}^\alpha$. Moreover, $S_\alpha(g_j)$ is an epimorphism by [Kra01, Theorem A], therefore $\coprod_{j\in J} Z_j$ is in $\mathscr{S}$. □

Proposition 8.4. *Let $\mathscr{T}$ be an α-compactly generated triangulated category and $\kappa>\alpha$ be regular cardinal such that either κ is strongly inaccessible or $2^\lambda=\lambda^+$ for every $\lambda<\kappa$. If* $\operatorname{card}\mathscr{T}^\alpha\leq\kappa$*, then* $\operatorname{card}\mathscr{T}^\kappa=\kappa$.

Proof. Our assumptions on κ and [Jec03, Theorem 5.20] show that $\kappa^{<\kappa}=\kappa$. Taking $\beta=\kappa^+$ in Theorem 8.1, we deduce that the size of morphism sets in $\mathscr{T}^{\kappa^+}$ is $<\kappa^+$, i.e. $\leq\kappa$. Hence the same is true for $\mathscr{T}^\kappa\subset\mathscr{T}^{\kappa^+}$.

By the proof of [Nee01b, Lemma 3.2.4 and Proposition 3.2.5], the set of objects S_κ of $\mathscr{T}^\kappa$ can be constructed as a continuous increasing union $S_\kappa=\bigcup_{\mu<\kappa}S_\mu$ starting with the set S_0 of objects of $\mathscr{T}^\alpha$. The set $S_{\mu+1}$ is defined from S_μ by adding coproducts of $<\kappa$ objects in S_μ and mapping cones of all possible morphisms between such coproducts. Assume that $\operatorname{card}S_\mu\leq\kappa$. Adding coproducts of $<\kappa$ objects increases the cardinal at most to $(\operatorname{card}S_\mu)^{<\kappa}\leq\kappa^{<\kappa}=\kappa$. Adding mapping cones neither increases the cardinal of S_μ since the size of morphism sets in $\mathscr{T}^\kappa$ is $\leq\kappa$. □

Corollary 8.5. *Let $\mathscr{T}$ be an $\aleph_1$-compactly generated triangulated category. Assuming the continuum hypothesis, if* $\operatorname{card}\mathscr{T}^{\aleph_0}\leq\aleph_1$*, then* $\operatorname{card}\mathscr{T}^{\aleph_1}=\aleph_1$.

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