

CELLULAR CATEGORIES

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ABSTRACT. We study locally presentable categories equipped with a cofibrantly generated weak factorization system. Our main result is that these categories are closed under 2-limits, in particular under pseudopullbacks. We give applications to deconstructible classes in Grothendieck categories. We discuss pseudopullbacks of combinatorial model categories.

1. INTRODUCTION

We introduce cellular categories as categories equipped with a class of morphisms containing all isomorphisms and closed under pushout and transfinite composite (= transfinite composition). The special case is a category equipped with a weak factorization system, which includes categories equipped with a factorization system. The latter categories are called “structured” in [1]. Cellular categories are abundant in homotopy theory because any Quillen model category carries two weak factorization systems, i.e., two cellular structures given by cofibrations and trivial cofibrations, resp. There are also various concepts of “cofibration categories” equipped with cofibrations and weak equivalences (see [17] for a recent survey). One can do homotopy theory in any category equipped with a weak factorization system because we have cylinder objects and hence homotopies there (see [12]). Cellular category does not need to have weak factorizations – for example pure monomorphisms in certain locally finitely presentable categories (see [8]). However, in a locally presentable category, one always has weak factorizations whenever cellular morphism are generated by a set of morphisms. The left part of the corresponding factorization system

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consists of retracts of cellular morphisms. In harmony with the J. Smith's concept of a combinatorial model category, we call such cellular categories combinatorial.

Our main result is that combinatorial cellular categories are closed under constructions of limit type. Like for locally presentable (or accessible categories) these limits should be defined in the framework of 2-categories and they can be reduced to products, inserters and equifiers (see [15] and [3]). These limits are called PIE-limits. The consequence is that they include both lax limits and pseudolimits. It turns out that the key step is the closedness under pseudopullbacks and the key ingredience is the use of good colimits introduced by Lurie [13] and futher developed by the first author in [14]. Lurie used good colimits for lifting cellular structure to functor categories, which is a limit type construction.

Our starting point is [16] and we are using notation from that paper. Among others, the present paper links combinatorial cellular categories with deconstructible classes of objects in Grothendieck abelian categories (see [10], [9], [20], [21] and [5]) where good colimits are replaced by generalized Hill lemma. Our limit theorem for combinatorial cellular categories implies some limit theorems for deconstructible classes proved in [20] and [5].

2. COMBINATORIAL CATEGORIES

Let \mathcal{X} be a class of morphisms in a category \mathcal{K} . Then $\text{Po}(\mathcal{X})$ denotes the class of pushouts of morphisms in \mathcal{X} : $f \in \text{Po}(\mathcal{X})$ iff f is an isomorphism or there is a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow & & \uparrow \\ X & \xrightarrow{g} & Y \end{array}$$

with $g \in \mathcal{X}$.

$\text{Tc}(\mathcal{X})$ denotes the class of transfinite composites (= compositions) of morphisms from \mathcal{X} : $f \in \text{Tc}(\mathcal{X})$ iff there is a smooth chain $(f_{ij}: A_i \rightarrow A_j)_{i \leq j \leq \lambda}$ (i.e., λ is an ordinal, $(f_{ij}: A_i \rightarrow A_j)_{i < j}$ is a colimit for any limit ordinal $j \leq \lambda$) such that $f_{i,i+1} \in \mathcal{X}$ for each $i < \lambda$ and $f = f_{0\lambda}$.

$\text{Rt}(\mathcal{X})$ denotes the class of retracts of morphisms in \mathcal{X} in the category \mathcal{K}^2 of morphisms of \mathcal{K} .

Definition 2.1. A cocomplete category \mathcal{K} is called *cellular* if it is equipped with a class \mathcal{C} of morphisms closed under pushout and transfinite composite.

Morphisms belonging to \mathcal{C} are called *cellular* and \mathcal{C} will be often denoted as $\text{cell}(\mathcal{K})$. Given a class \mathcal{X} of morphisms of a cocomplete category \mathcal{K} then $\text{cell}(\mathcal{X})$ denotes the closure of \mathcal{X} under pushout and transfinite composite. In fact, $\text{cell}(\mathcal{X})$ consists of transfinite composites of pushouts of morphisms from \mathcal{X} . We say that the cellular category $(\mathcal{K}, \text{cell}(\mathcal{X}))$ is *cellularly generated* by \mathcal{X} . In a cellular category \mathcal{K} , let

$$\text{cof}(\mathcal{K}) = \text{Rt cell}(\mathcal{K})$$

Elements of this class are called *cofibrations*.

Lemma 2.2. *Let \mathcal{K} be a cellular category. Then $(\mathcal{K}, \text{cof}(\mathcal{K}))$ is a cellular category.*

Proof. It is easy to see that $\text{cof}(\mathcal{K})$ is closed under pushout. Let $f_0 : A_0 \rightarrow A_1$ and $f_1 : A_1 \rightarrow A_2$ be two composable cofibrations. Following [16] 2.1(5), there are cellular morphisms $g_0 : A_0 \rightarrow B_1$, $h_1 : A_1 \rightarrow C_2$ and morphisms $u_1 : A_1 \rightarrow B_1$, $r_1 : B_1 \rightarrow A_1$, $v_2 : A_2 \rightarrow C_2$, $s_2 : C_2 \rightarrow A_2$ such that u_1, r_1 make f_0 a retract of g_0 in $A_0 \setminus \mathcal{K}$ and v_2, s_2 make f_1 a retract of h_1 in $A_1 \setminus \mathcal{K}$. Consider a pushout

$$\begin{array}{ccc} B_1 & \xrightarrow{g_1} & B_2 \\ \uparrow u_1 & & \uparrow u_2 \\ A_1 & \xrightarrow{f_1} & A_2 \end{array}$$

Since $h_1 r_1 u_1 = h_1 = v_2 f_1$, there is the unique morphism $t : B_2 \rightarrow C_2$ such that $t g_1 = h_1 r_1$ and $t u_2 = v_2$. It is easy to see that u_2 and $s_2 t$ make f_1 a retract of g_1 . Thus $f_1 f_0$ is a retract of $g_1 g_0$.

Let $(f_{ij} : A_i \rightarrow A_j)_{i \leq j \leq \lambda}$ be a smooth chain of cofibrations. By recursion, we construct a smooth chain of cellular morphisms $(g_{ij} : B_i \rightarrow B_j)_{i \leq j \leq \lambda}$ such that each f_{0i} is a retract of g_{0i} . In isolated steps we proceed as above and in limits steps we take colimits. \square

We say that a cellular category \mathcal{K} is *retract closed* if $\text{cof}(\mathcal{K}) = \text{cell}(\mathcal{K})$. Following 2.2, $(\mathcal{K}, \text{cof}(\mathcal{K}))$ is a reflection of a cellular category \mathcal{K} into retract closed cellular categories.

Given a set of morphisms in a cocomplete category \mathcal{K} then $\text{cof}(\mathcal{X})$ denotes the closure of \mathcal{X} under pushout, transfinite composite and retract. In fact, $\text{cof}(\mathcal{X})$ consists of retracts of transfinite composites of pushouts of morphisms from \mathcal{X} . We say that $\text{cof}(\mathcal{X})$ is *cofibrantly generated* by \mathcal{X} .

Definition 2.3. A retract closed cellular category \mathcal{K} is called *combinatorial* if \mathcal{K} is locally presentable and $\text{cof}(\mathcal{K})$ is cofibrantly generated by a set of morphisms from \mathcal{K} .

In a combinatorial category \mathcal{K} , the class of cofibrations forms a left part of a weak factorization system. Following [13] A.1.5.12, any combinatorial category is cellularly generated by a set of morphisms.

An object K is called *cofibrant* if the unique morphism $0 \rightarrow K$ from an initial object is a cofibration. Analogously we define *cellular* objects.

Example 2.4. Any cocomplete category carries two cellular structures – the *discrete* $\mathcal{K}_d = (\mathcal{K}, \text{Iso})$ and the *trivial* $\mathcal{K}_t = (\mathcal{K}, \mathcal{K}^2)$. They are both retract closed. Moreover, if \mathcal{K} is locally presentable, they are both combinatorial. This is evident for the discrete one. Let \mathcal{K} be a locally κ -presentable category. Then the trivial structure is combinatorial because \mathcal{K}^2 is cofibrantly generated by morphisms between κ -presentable objects. (see [18], 4.6 for $\kappa = \omega$ but the general case is the same).

A morphism $(u, v) : g \rightarrow f$ in \mathcal{K}^2 will be called a *pushout morphism* if the square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow u & & \uparrow v \\ C & \xrightarrow{g} & D \end{array}$$

is a pushouts. Let $\text{psh}(\mathcal{K}^2)$ be the subcategory of \mathcal{K}^2 with the objects as \mathcal{K}^2 and with the pushout morphisms.

Let \mathcal{K}_κ denote the full subcategory of \mathcal{K} consisting of κ -presentable objects and $(\mathcal{K}_\kappa)^2$ the category of morphisms of \mathcal{K}_κ . For a class $\mathcal{X} \subseteq (\mathcal{K}_\kappa)^2$, we put

$$\text{Po}_\kappa(\mathcal{X}) = \text{Po}(\mathcal{X}) \cap (\mathcal{K}_\kappa)^2$$

$$\text{cell}_\kappa(\mathcal{X}) = \text{cell}(\mathcal{X}) \cap (\mathcal{K}_\kappa)^2$$

$$\text{cof}_\kappa(\mathcal{X}) = \text{cof}(\mathcal{X}) \cap (\mathcal{K}_\kappa)^2$$

and denote by $\kappa\text{-Tc}(\mathcal{X})$ the class of transfinite composites of length smaller than κ .

Lemma 2.5. *Let \mathcal{K} be a locally κ -presentable category and \mathcal{X} a set of morphisms between κ -presentable objects. Then the full subcategory of $\text{psh}(\mathcal{K}^2)$ on objects belonging to $\text{Po}(\mathcal{X})$ is κ -accessible with $\text{Po}_\kappa(\mathcal{X})$ being the full subcategory of κ -presentable objects.*

Proof. Our category clearly has κ -directed colimits calculated in \mathcal{K}^2 and objects from $\text{Po}_\kappa(\mathcal{X})$ are κ -presentable. Thus it suffices to prove that any morphism in $\text{Po}(\mathcal{X})$ is a κ -directed colimit in $\text{psh}(\mathcal{K}^2)$ of morphisms belonging to $\text{Po}_\kappa(\mathcal{X})$.

Let $f : A \rightarrow B$ be a morphism in $\text{Po}(\mathcal{X})$. Thus there is a pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow u & & \uparrow v \\ X & \xrightarrow{g} & Y \end{array}$$

where $g \in \mathcal{X}$. We can express A as a κ -directed colimit $(h_i : A_i \rightarrow A)_{i \in I}$ of κ -presentable objects. Let $(h_{ij} : A_i \rightarrow A_j)_{i \leq j \in I}$ denote the corresponding diagram. Since X is κ -presentable, there is a factorization

$$u : X \xrightarrow{u_i} A_i \xrightarrow{h_i} A$$

for some $i \in I$. Let $u_j = h_{ij}u_i$ for $i \leq j \in I$. Form pushouts

$$\begin{array}{ccc} A_j & \xrightarrow{g_j} & B_j \\ \uparrow u_j & & \uparrow v_j \\ X & \xrightarrow{g} & Y \end{array}$$

We get the diagram

$$\begin{array}{ccccc} & & A & \xrightarrow{f} & B \\ & & \uparrow h_j & & \uparrow k_j \\ & & A_j & \xrightarrow{g_j} & B_j \\ & & \uparrow u_j & & \uparrow v_j \\ X & \xrightarrow{g} & Y & & \end{array}$$

where the lower square and the outer rectangle are pushouts and k_j is the induced morphism. Thus the upper square is the pushout. Hence f is a κ -directed colimit in $\text{psh}(\mathcal{K}^2)$ of morphisms $g_j : A_j \rightarrow B_j$ belonging to $\text{Po}_\kappa(\mathcal{X})$. \square

3. LIMITS OF COMBINATORIAL CATEGORIES

A functor $F : \mathcal{K} \rightarrow \mathcal{L}$ between cellular categories will be called *cellular* if it preserves colimits and cellular morphisms. We will denote **CAT** the 2-category of categories, functors and natural transformations and **CELL** the 2-category of cellular categories, cellular functors and natural transformations. The forgetful 2-functor $U : \mathbf{CELL} \rightarrow \mathbf{CAT}$ has both a left 2-adjoint given by discrete cellular structures and a right 2-adjoint given by trivial ones. In particular, U preserves all existing 2-limits. We are not really interested in 2-limits in **CELL** but, for what follows, it is instructive to calculate pseudopullbacks.

We recall that a pseudopullback of functors F and G is a square in **CAT**

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\bar{F}} & \mathcal{L} \\ \bar{G} \downarrow & & \downarrow G \\ \mathcal{K} & \xrightarrow{F} & \mathcal{M} \end{array}$$

which commutes up to an isomorphism and has the 2-categorical universal property among such squares. Objects of the category \mathcal{P} are triples (K, L, t) where $t : FK \rightarrow GL$ is an isomorphism and morphisms $(K_1, L_1, t_1) \rightarrow (K_2, L_2, t_2)$ are pairs (u, v) where $u : K_1 \rightarrow K_2$, $v : L_1 \rightarrow L_2$ such that $t_2 F(u) = G(v)t_1$. The functors \bar{F} , \bar{G} are the projections and t 's yield the desired natural isomorphism $F\bar{G} \rightarrow G\bar{F}$.

Given $\mathcal{C} \subseteq \mathcal{K}^2$ and $\mathcal{D} \subseteq \mathcal{L}^2$, we get the class

$$\text{Ps}(\mathcal{C}, \mathcal{D}) = \{f \in \mathcal{P}^2 \mid \bar{F}f \in \mathcal{D}, \bar{G}f \in \mathcal{C}\}$$

of morphisms in \mathcal{P} .

Lemma 3.1. **CELL** has pseudopullbacks.

Proof. Let $F : \mathcal{K} \rightarrow \mathcal{M}$ and $G : \mathcal{L} \rightarrow \mathcal{M}$ be cellular functors. It suffices to put

$$\text{cell}(\mathcal{P}) = \text{Ps}(\text{cell}(\mathcal{K}), \text{cell}(\mathcal{L})).$$

□

We will denote **LOC** the 2-category of locally presentable categories, colimit preserving functors and natural transformations. Recall that any colimit preserving functor between locally presentable categories has a left adjoint. **LOC** has all PIE-limits, which means products, inserters and equifiers. Consequently, it has all pseudolimits, in particular it has pseudopullbacks. This basic result was proved in [6] and

follows from a more general limit theorem for accessible categories (see [15]) where one can find all needed concepts (see also [3]).

A functor $F : \mathcal{K} \rightarrow \mathcal{L}$ between combinatorial categories will be called *combinatorial* if it preserves colimits and cofibrations. **COMB** will denote the 2-category of combinatorial categories, combinatorial functors and natural transformations. Again, the forgetful 2-functor $V : \mathbf{COMB} \rightarrow \mathbf{LOC}$ has both a left 2-adjoint and right 2-adjoint given by discrete and trivial combinatorial structures. Thus V preserves all existing 2-limits. Moreover **COMB** is a full sub-2-category of **CELL**. Let us add that, unlike **CAT** and **CELL**, there are no set-theoretical difficulties with the categories **LOC** and **COMB** because any morphism is determined by its restriction on a small part.

Recall that a poset P is well-founded if every of its nonempty subsets contains a minimal element. A well-founded poset with a least element \perp is called *good*. Any well-ordered set is good. An element x of a good poset P is called *isolated* if

$$\Downarrow x = \{y \in P \mid y < x\}$$

has a top element x^- which is called the *predecessor* of x . A non-isolated element distinct from \perp is called *limit*. Given $x < y$ in a poset P , we denote xy the unique morphism $x \rightarrow y$ in the category P .

A diagram $D: P \rightarrow \mathcal{K}$ is *smooth* if, for every limit $x \in P$, the diagram $(D(yx): Dy \rightarrow Dx)_{y < x}$ is a colimit cocone on the restriction of D to $\Downarrow x$. A *good* diagram $D: P \rightarrow \mathcal{K}$ is a smooth diagram whose shape category P is a good poset.

A good poset is called κ -*good* if all its initial segments $\downarrow x = \{y \in P \mid y \leq x\}$ have cardinality $< \kappa$. All these concepts can be found in [16].

Theorem 3.2. **COMB** has pseudopullbacks calculated in **CELL**.

Proof. Consider a pseudopullback in **CELL**

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\bar{F}} & \mathcal{L} \\ \bar{G} \downarrow & & \downarrow G \\ \mathcal{K} & \xrightarrow{F} & \mathcal{M} \end{array}$$

where F and G are combinatorial functors. We have to show that \mathcal{P} is combinatorial, i.e., that $\text{Ps}(\text{cof}(\mathcal{K}), \text{cof}(\mathcal{L}))$ is cofibrantly generated by a set of morphisms.

There is an uncountable regular cardinal κ such that the categories $\mathcal{K}, \mathcal{L}, \mathcal{M}$ are locally κ -presentable, both $\text{cof}(\mathcal{K})$ and $\text{cof}(\mathcal{L})$ are cofibrantly generated by morphisms between κ -presentable objects, \mathcal{P} is locally κ -presentable and the functors $F, G, \overline{F}, \overline{G}$ preserve κ -filtered colimits and κ -presentable objects. Following [13] A.1.5.12, both $\text{cof}(\mathcal{K})$ and $\text{cof}(\mathcal{L})$ are cellularly generated by morphisms between κ -presentable objects. Let \mathcal{X} consist of morphisms f between κ -presentable objects in \mathcal{P} such that $\overline{F}f \in \text{cof}(\mathcal{L})$ and $\overline{G}f \in \text{cof}(\mathcal{K})$. We will prove that

$$\text{cell}(\mathcal{X}) = \text{Ps}(\text{cof}(\mathcal{K}), \text{cof}(\mathcal{L})),$$

which proves the theorem. Recall that $\text{cof}_\kappa(\mathcal{K})$ denotes cofibrations between κ -presentable objects in \mathcal{K} and the same for $\text{cof}_\kappa(\mathcal{L})$. Then

$$\mathcal{X} = \text{Ps}(\text{cof}_\kappa(\mathcal{K}), \text{cof}_\kappa(\mathcal{L})).$$

Thus it suffices to prove the equations

$$\begin{aligned} \text{Ps}(\text{cof}(\mathcal{K}), \text{cof}(\mathcal{L})) &= \text{Tc Ps}(\text{Po cof}_\kappa(\mathcal{K}), \text{Po cof}_\kappa(\mathcal{L})) \\ &= \text{Tc Po Ps}(\text{cof}_\kappa(\mathcal{K}), \text{cof}_\kappa(\mathcal{L})). \end{aligned}$$

In the first equation, the right-hand side is obviously included in the left-hand side. Let e belong to the left-hand side of the first equation. Following [16] 4.11, there are κ -good κ -directed diagrams $D : P \rightarrow \mathcal{K}$ and $E : Q \rightarrow \mathcal{L}$ with links in $\text{Po cof}_\kappa(\mathcal{K})$ and $\text{Po cof}_\kappa(\mathcal{L})$ resp. such that $\overline{F}e$ is the composite of E and $\overline{G}e$ is the composite of D . Thus there are isomorphisms u and v in \mathcal{M} such that the square

$$\begin{array}{ccc} FD \perp & \xrightarrow{F\overline{G}e} & F \text{colim } D \\ \downarrow u & & \downarrow v \\ GE \perp & \xrightarrow{G\overline{F}e} & G \text{colim } E \end{array}$$

commutes. In what follows, $\delta : D \rightarrow \text{colim } D$ and $\varepsilon : E \rightarrow \text{colim } E$ are colimit cocones. We can assume that neither P nor Q have the greatest element.

Let \tilde{P} denote the set of all non-empty initial segments X of P . For each $X \in \tilde{P}$, we get the induced morphism $\delta_X : \text{colim } D_X \rightarrow \text{colim } D$ where D_X denotes the restriction of D on X . Analogously, we have $\varepsilon_Y : \text{colim } E_Y \rightarrow \text{colim } E$ for $Y \in \tilde{Q}$. Given $X \in \tilde{P}$ and $x \in P$, let $X(x) = X \cup \downarrow x$. Since P is directed and does not have the greatest element, there is $x < y \in P$. Clearly, either $y \notin X(x)$ or $X(x) = X$. Thus $X \neq P$ implies $X(x) \neq P$.

By recursion on all ordinals i and j , $i < j$, we will construct smooth chains $X_i \subseteq X_j$ in \tilde{P} , $Y_i \subseteq Y_j$ in \tilde{Q} and

$$(k_{ij}, l_{ij}) : (K_i, L_i, u_i) \rightarrow (K_j, L_j, u_j)$$

in \mathcal{P} such that each K_i is a colimit of the restriction of D on an initial segment $X_i \in \tilde{P}$, each L_i is a colimit of the restriction of E on an initial segment $Y_i \in \tilde{Q}$, $k_{ij} : K_i \rightarrow K_j$, $l_{ij} : L_i \rightarrow L_j$ are the induced morphisms, and, for the induced morphisms $k_i : K_i \rightarrow \text{colim } D$, $l_i : L_i \rightarrow \text{colim } E$, the square

$$\begin{array}{ccc} FK_i & \xrightarrow{Fk_i} & F \text{ colim } D \\ \downarrow u_i & & \downarrow v \\ GL_i & \xrightarrow{Gl_i} & G \text{ colim } E \end{array}$$

commutes. The construction will terminate at the ordinal k when both $X_k = P$ and $Y_k = Q$ become true. It follows easily from [16] 4.20 that each morphism k_{ij} is in $\text{Po cof}_\kappa(\mathcal{K})$, and similarly $l_{ij} \in \text{Po cof}_\kappa(\mathcal{L})$, thus the smooth chain $(k_{ij}, l_{ij})_{i \leq j < k}$ has links in $\text{Ps}(\text{Po cof}_\kappa(\mathcal{K}), \text{Po cof}_\kappa(\mathcal{L}))$. In this way we get that e belongs to the right-hand side of the first equation.

We put $K_0 = D \perp$, $L_0 = E \perp$ and $u_0 = u$. Let us have (K_i, L_i, u_i) . If $X_i = P$ and $Y_i = Q$, we are finished, and we put $k = i$. Otherwise, either $Y_i \neq Q$, or $X_i \neq P$. Assume the first case, the second is handled symmetrically. Choose $y_1 \in Q - Y_i$ and put $Y_{i1} = Y_i(y_1)$, $L_{i1} = \text{colim } E_{Y_{i1}}$. Following [16] 4.20, the induced morphism $s_0 : L_i \rightarrow L_{i1}$ belongs to $\text{Po cof}_\kappa(\mathcal{L})$. Let

$$\begin{array}{ccc} L_i & \xrightarrow{s_0} & L_{i1} \\ \uparrow a & & \uparrow b \\ A & \xrightarrow{h} & B \end{array}$$

be a corresponding pushout with $h \in \text{cof}_\kappa(\mathcal{L})$. Since GB is κ -presentable and P κ -directed, there is $x_0 \in P$ and

$$f : GB \rightarrow FDx_0$$

such that $v^{-1}G(\varepsilon_{Y_{i1}})G(b) = F(\delta_{x_0})f$. We obtain the morphisms

$$f_1 : GB \xrightarrow{f} FDx_0 \rightarrow F \text{ colim } D_{X_i(x_0)}$$

and

$$f_2 : GL_i \xrightarrow{u_i^{-1}} FK_i \rightarrow F \operatorname{colim} D_{X_i(x_0)}$$

Since

$$F\delta_{X_i(x_0)}f_1G(h) = F\delta_{X_i(x_0)}f_2G(a),$$

$F\delta_{X(x)} : F \operatorname{colim} D_{X(x)} \rightarrow F \operatorname{colim} D$ is a κ -directed colimit cocone and GA is κ -presentable, there is $x_0 \leq x_1 \in P$ such that the morphisms

$$g_1 : GB \xrightarrow{f_1} F \operatorname{colim} D_{X_i(x_0)} \rightarrow F \operatorname{colim} D_{X_i(x_1)}$$

and

$$g_2 : GL_i \xrightarrow{f_2} F \operatorname{colim} D_{X_i(x_0)} \rightarrow F \operatorname{colim} D_{X_i(x_1)}$$

satisfy $g_1G(h) = g_2G(a)$. We put $X_{i1} = X(x_1)$, $K_{i1} = \operatorname{colim} D_{X_{i1}}$ and $u_{i1} : GL_{i1} \rightarrow FK_{i1}$ is the induced morphism from the pushout defining GL_{i1} . Following [16] 4.20, the induced morphism $t_0 : K_i \rightarrow K_{i1}$ belongs to $\operatorname{Po} \operatorname{cof}_\kappa(\mathcal{K})$. We have $u_{i1}G(s_0)u_i = F(t_0)$ and $v^{-1}G(\varepsilon_{Y_{i1}}) = F(\delta_{X_{i1}})u_{i1}$.

By applying the same argument to t_0 at the place of s_0 , we get $x_1 \leq x_2$ in P , $y_1 \leq y_2$ in Q , initial segments $X_{i2} = X(x_2) \in \tilde{P}$ and $Y_{i2} = Y(y_2) \in \tilde{Q}$, the objects $K_{i2} = \operatorname{colim} D_{X_{i2}}$ and $L_{i2} = \operatorname{colim} E_{Y_{i2}}$ and morphisms $t_1 : K_{i1} \rightarrow K_{i2}$ in $\operatorname{Po} \operatorname{cof}_\kappa(\mathcal{K})$, $s_1 : L_{i1} \rightarrow L_{i2}$ in $\operatorname{Po} \operatorname{cof}_\kappa(\mathcal{L})$ and $u_{i2} : FK_{i2} \rightarrow GL_{i2}$ such that $vF(\delta_{X_{i2}}) = G(\varepsilon_{Y_{i2}})u_{i2}$ and $u_{i2}F(t_1)u_{i1} = G(s_1)$. We have

$$u_{i2}F(t_1t_0) = u_{i2}F(t_1)u_{i1}G(s_0)u_i = G(s_1s_0)u_i.$$

Continuing this procedure, we get morphisms u_{in} with alternating directions, whose squares with v or v^{-1} commute and all squares between u_{in} 's for odd n and between u_{in} 's for even n commute. We put $X_{i+1} = \cup X_{in}$, $Y_{i+1} = \cup Y_{in}$, $K_{i+1} = \operatorname{colim} D_{X_{i+1}}$, $L_{i+1} = \operatorname{colim} E_{Y_{i+1}}$ and $u_{i+1} = \operatorname{colim} u_{i(2n)}$. Clearly, u_{i+1} is an isomorphism; its inverse is $\operatorname{colim} u_{i(2n+1)}$.

The construction of the items at stage i for i a limit ordinal is dictated by the smoothness requirements. Clearly, there is an ordinal k where the construction stops: $X_k = P$ and $Y_k = Q$. Since now f_k is the identity on $\operatorname{colim} D$, and similarly for g_k , it follows that $u_k = v$. Thus e is the composite of the diagram $(k_{ij}, l_{ij})_{i < k \leq k}$ as desired.

The second equation is the consequence of

$$\operatorname{Ps}(\operatorname{Po} \operatorname{cof}_\kappa(\mathcal{K}), \operatorname{Po} \operatorname{cof}_\kappa(\mathcal{L})) = \operatorname{Po} \operatorname{Ps}(\operatorname{cof}_\kappa(\mathcal{K}), \operatorname{cof}_\kappa(\mathcal{L})).$$

In this equation, the right-hand side is obviously contained in the left-hand side. Let e belong to the left-hand side. Then $\overline{F}e : L_1 \rightarrow L_2$ is a pushout of a morphism from $\operatorname{cof}_\kappa(\mathcal{L})$ and $\overline{G}e : K_1 \rightarrow K_2$ is a pushout

of a morphism from $\text{cof}_\kappa(\mathcal{K})$. There are isomorphisms u and v in \mathcal{M} such that the square

$$\begin{array}{ccc} FK_1 & \xrightarrow{F\overline{G}e} & FK_2 \\ \downarrow u & & \downarrow v \\ GL_1 & \xrightarrow{G\overline{F}e} & GL_2 \end{array}$$

commutes.

Following 2.5, $\overline{G}e$ is a κ -directed colimit in $\text{psh}(\mathcal{K}^2)$ of morphisms from $\text{cof}_\kappa(\mathcal{K})$ and $\overline{F}e$ is a κ -directed colimit in $\text{psh}(\mathcal{L}^2)$ of morphisms from $\text{cof}_\kappa(\mathcal{L})$. Let $g_0 : L_{10} \rightarrow L_{20}$ be a morphism from $\text{cof}_\kappa(\mathcal{L})$, $(l_{10}, l_{20}) : g_0 \rightarrow \overline{F}e$ a morphism in $\text{psh}(\mathcal{L}^2)$, $f_0 : K_{10} \rightarrow K_{20}$ a morphism from $\text{cof}_\kappa(\mathcal{K})$ and $(k_{10}, k_{20}) : f_0 \rightarrow \overline{G}e$ a morphism in $\text{psh}(\mathcal{K}^2)$. Since f_0 is κ -presentable in $\text{psh}(\mathcal{K}^2)$ and thus Ff_0 is κ -presentable in $\text{psh}(\mathcal{M}^2)$, there is a morphism $g_1 : L_{11} \rightarrow L_{21}$ from $\text{cof}_\kappa(\mathcal{L})$ and a morphism $(l_{11}, l_{21}) : g_1 \rightarrow \overline{F}e$ with factorizations

$$(l_{10}, l_{20}) = (l_{11}, l_{21}) \cdot (l_{101}, l_{201})$$

and

$$(u, v) \cdot (Fk_{10}, Fk_{20}) = (Gl_{11}, Gl_{21})(u_1, v_1).$$

Analogously, there is a morphism $f_1 : K_{11} \rightarrow K_{21}$ from $\text{cof}_\kappa(\mathcal{K})$ and a morphism $(k_{11}, k_{21}) : f_1 \rightarrow \overline{G}e$ with factorizations

$$(k_{10}, k_{20}) = (k_{11}, k_{21}) \cdot (k_{101}, k_{201})$$

and

$$(u^{-1}, v^{-1}) \cdot (Gl_{11}, Gl_{21}) = (Fk_{11}, Fk_{21})(u_2, v_2).$$

We continue this procedure and take colimits of the resulting chains $g = \text{colim } g_n$ and $f = \text{colim } f_n$. Since Gg and Ff are isomorphic, we get a morphism e' from $\text{Ps}(\text{cof}_\kappa(\mathcal{K}), \text{cof}_\kappa(\mathcal{L}))$ and a pushout morphism $e' \rightarrow e$. Therefore e belongs to $\text{Po Ps}(\text{cof}_\kappa(\mathcal{K}), \text{cof}_\kappa(\mathcal{L}))$. \square

Corollary 3.3. **COMB** has PIE-limits calculated in **CELL**.

Proof. Products of combinatorial categories are evident:

$$\prod_{i \in I} (\mathcal{K}_i, \mathcal{X}_i) = \left(\prod_{i \in I} \mathcal{K}_i, \prod_{i \in I} \mathcal{X}_i \right).$$

Let $F, G : \mathcal{K} \rightarrow \mathcal{L}$ be combinatorial functors, $\varphi, \psi : F \rightarrow G$ natural transformations and $\text{Eq}(\varphi, \psi)$ their equifier in **CAT**. Consider a

pullback

$$\begin{array}{ccc}
 \mathcal{P} & \longrightarrow & \mathcal{K} \\
 \downarrow & & \downarrow \text{Id} \\
 \text{Eq}(\varphi, \psi)_t & \xrightarrow{V} & \mathcal{K}_t
 \end{array}$$

where $V : \text{Eq}(\varphi, \psi) \rightarrow \mathcal{K}$ is the full embedding. Since Id is transportable, this pullback is equivalent to a pseudopullback (see [15], 5.1.1). Since any category equivalent to a combinatorial category is combinatorial, \mathcal{P} is combinatorial and is an equifier of φ and ψ in **COMB**.

Finally, let $F, G : \mathcal{K} \rightarrow \mathcal{L}$ be combinatorial functors and $\text{Ins}(F, G)$ their inserter in **CAT**. Consider a pullback

$$\begin{array}{ccc}
 \mathcal{P} & \longrightarrow & \mathcal{K} \\
 \downarrow & & \downarrow \text{Id} \\
 \text{Ins}(F, G)_t & \xrightarrow{V} & \mathcal{K}_t
 \end{array}$$

where $V : \text{Ins}(F, G) \rightarrow \mathcal{K}$ is the forgetful functor. Analogously as above, \mathcal{P} is an inserter of F and G in **COMB**.

Clearly, all PIE-limits above are calculated in **CELL**. \square

The consequence is that **COMB** has pseudolimits and lax limits (calculated in **CAT**). The same is true for **CELL**. Another consequence is [13] 2.8.3 (see [14] as well).

Corollary 3.4. *Let \mathcal{K} be a combinatorial category and \mathcal{C} a small category. Then the functor category $\mathcal{K}^{\mathcal{C}}$ is combinatorial with respect to the pointwise combinatorial structure.*

Proof. The cotensor $[\mathcal{C}, \mathcal{K}]$ taken in **CELL** is the cellular category described in the Corollary. Since $[\mathcal{C}, \mathcal{K}]$ can be constructed using PIE-limits, 3.3 implies that $[\mathcal{C}, \mathcal{K}]$ is in **COMB** provided that \mathcal{K} is in **COMB**. \square

Remark 3.5. Let \mathcal{K} be a Grothendieck abelian category and \mathcal{C} a class of objects in \mathcal{K} . A \mathcal{C} -monomorphism is a monomorphism whose cokernel belongs to \mathcal{C} . The class \mathcal{C} -Mono of these monomorphisms makes \mathcal{K} a cellular category. Cellular objects here are precisely \mathcal{C} -filtered objects, i.e., objects K such that the morphism $0 \rightarrow K$ is a transfinite composite of \mathcal{C} -monomorphisms. A class \mathcal{C} is deconstructible if it is

a class of \mathcal{S} -filtered objects for a set \mathcal{S} . The fundamental fact (basically due to [21]) is that \mathcal{C} is deconstructible if and only if the cellular category $(\mathcal{K}, \mathcal{C}\text{-Mono})$ is combinatorial.

Let \mathcal{C} be deconstructible and $\mathbf{C}(\mathcal{K})$ be the category of complexes over \mathcal{K} . Since it is a functor category, 3.4 implies that $\mathbf{C}(\mathcal{K})$ is combinatorial with respect to pointwise \mathcal{C} -monomorphisms. Consequently the class $\mathbf{C}(\mathcal{C})$ of complexes with components in \mathcal{C} is deconstructible, which was proved in [20] 4.2 (1) using generalized Hill lemma.

Let $T : \mathcal{K} \rightarrow \mathcal{K}$ be a colimit preserving monad on a combinatorial category \mathcal{K} and $U : \text{Alg}(T) \rightarrow \mathcal{K}$ the forgetful functor from the category of T -algebras. Then $\text{Alg}(T)$ is locally presentable (see [3] Remark 2.78) and U preserves colimits (see [7] 4.3.2). Thus $\text{Alg}(T)$ is a cellular category where f is a cofibration if and only if Uf is a cofibration.

Corollary 3.6. *Let $T : \mathcal{K} \rightarrow \mathcal{K}$ be a colimit preserving monad on a combinatorial category \mathcal{K} . Then $\text{Alg}(T)$ is combinatorial.*

Proof. The combinatorial category $\text{Alg}(T)$ is given by a pseudopullback

$$\begin{array}{ccc} \text{Alg}(T) & \xrightarrow{U} & \mathcal{K} \\ \text{Id} \downarrow & & \downarrow \text{Id} \\ \text{Alg}(T)_t & \xrightarrow{U} & \mathcal{K}_t \end{array}$$

□

Remark 3.7. Let $T : \mathcal{K} \rightarrow \mathcal{K}$ be a colimit preserving monad on a Grothendieck abelian category and \mathcal{C} a deconstructible class of objects in \mathcal{K} . Then the class of T -algebras A with $UA \in \mathcal{C}$ is deconstructible in $T\text{-Alg}$. This result was proved in [5] A.7 and follows from 3.6 and 3.5.

Remark 3.8. More generally, let $F : \mathcal{K} \rightarrow \mathcal{L}$ be a colimit preserving functor from a locally presentable category \mathcal{K} to a combinatorial category \mathcal{L} . In the same way as above, we get a combinatorial structure on \mathcal{K} where f is a cofibration if and only if Ff is a cofibration. We have

a pseudopullback

$$\begin{array}{ccc}
 \mathcal{K} & \xrightarrow{F} & \mathcal{L} \\
 \text{Id} \downarrow & & \downarrow \text{Id} \\
 \mathcal{K}_t & \xrightarrow{F} & \mathcal{L}_t
 \end{array}$$

In accordance with [11] 4.1, we call this combinatorial structure *left-induced* from \mathcal{L} .

Let us observe that both equifiers and inserters are given as left-induced structures. Since PIE-limits yield all pseudolimits, we could prove only the special case of 3.2 giving the existence of left-induced structures.

4. LIMITS OF COMBINATORIAL MODEL CATEGORIES

Any combinatorial model category \mathcal{K} has two underlying combinatorial categories $W_1(\mathcal{K}) = (\mathcal{K}, \mathcal{C})$ and $W_2(\mathcal{K}) = (\mathcal{K}, \mathcal{C}_0)$ where \mathcal{C} is the class of cofibrations of \mathcal{K} and \mathcal{C}_0 is the class of trivial cofibrations. On every locally presentable category \mathcal{K} there is a *trivial* combinatorial model structure \mathcal{K}_{tm} such that both $W_1(\mathcal{K}_{tm})$ and $W_2(\mathcal{K}_{tm})$ are trivial combinatorial categories and a *discrete* combinatorial model structure \mathcal{K}_{dm} such that both $W_1(\mathcal{K}_{dm})$ and $W_2(\mathcal{K}_{dm})$ are discrete combinatorial categories. Weak equivalences are all morphisms in the both cases. More generally, any combinatorial category \mathcal{K} yields a combinatorial model category $m(\mathcal{K})$ such that $W_i(m(\mathcal{K})) = \mathcal{K}$ for $i = 1, 2$. Again, any morphism of \mathcal{K} is a weak equivalence in $m(\mathcal{K})$. In particular, $\mathcal{K}_{tm} = m(\mathcal{K}_t)$ and $\mathcal{K}_{dm} = m(\mathcal{K}_d)$.

Let **CMOD** denote the category of combinatorial model categories and left Quillen functors. We get the functors

$$W_1, W_2 : \mathbf{CMOD} \rightarrow \mathbf{COMB}.$$

Lemma 4.1. W_2 preserves pseudopullbacks existing in **CMOD**.

Proof. It follows from the fact that $m(-) : \mathbf{COMB} \rightarrow \mathbf{CMOD}$ is left adjoint to W_2 . □

We know that

$$\text{Ps}(\text{tcof}(\mathcal{K}), \text{tcof}(\mathcal{L})) \subseteq \text{cof}(\mathcal{P}) \subseteq \text{Ps}(\text{cof}(\mathcal{K}), \text{cof}(\mathcal{L})).$$

We will show that W_1 does not need to preserve existing pseudopullbacks.

Example 4.2. Let \mathcal{K} be the standard model category of simplicial sets. Let $t : 0 \rightarrow 1$ and \mathcal{L} be the model structure on simplicial sets where $\text{cof}(\{t\})$ is the class of cofibrations and any morphism is a weak equivalence. It is easy to see that cofibrations are precisely coproduct injections $K \rightarrow K \amalg D$ where K is a simplicial set and D is a discrete simplicial set. We will show that the discrete model structure \mathcal{K}_d on simplicial sets yields a pseudopullback

$$\begin{array}{ccc} \mathcal{K}_d & \xrightarrow{\text{Id}} & \mathcal{L} \\ \text{Id} \downarrow & & \downarrow \text{Id} \\ \mathcal{K} & \xrightarrow{\text{Id}} & \mathcal{K}_{tm} \end{array}$$

in **COMB**. Consider a model structure \mathcal{P} on simplicial sets such that $\text{Id} : \mathcal{P} \rightarrow \mathcal{K}$ and $\text{Id} : \mathcal{P} \rightarrow \mathcal{L}$ are left Quillen functors. Since $\text{cof}(\{t\})$ is the class of trivial cofibrations in \mathcal{L} , the intersection of trivial cofibrations in \mathcal{K} and \mathcal{L} contains only isomorphisms. Thus $\text{tcof}(\mathcal{P}) = \text{Iso}$. Since $\text{cof}(\{t\})$ is the intersection of cofibrations in \mathcal{K} and \mathcal{L} , trivial fibrations in \mathcal{P} should contain $\text{cof}(\{t\})^\square$, which is the class of surjective simplicial maps. Since trivial cofibrations in \mathcal{P} are isomorphisms, $\text{cof}(\mathcal{P})^\square$ is the class of weak equivalences in \mathcal{P} . Hence $\text{cof}(\mathcal{P})^\square$ has the 2-out-of-3 property and thus it contains all coproduct injections $u : A \rightarrow A \amalg B$. The reason that, for $B \neq \emptyset$, there is always $f : A \amalg B \rightarrow B$ with $f u = \text{id}_A$. Since $\text{cof}(\mathcal{P}) \cap \text{cof}(\mathcal{P})^\square = \text{Iso}$ (see [2] III.4 (2)), $\text{cof}(\mathcal{P}) = \text{Iso}$.

We do not know whether **CMOD** has pseudopullbacks.

Remark 4.3. (1) The existence of pseudopullbacks would imply the existence of PIE-limits in **CMOD**. Since products of combinatorial model categories exist and are preserved by W_1 and W_2 , PIE-limits can be obtained from pseudopullbacks of the kind

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\text{Id}} & \mathcal{K}_{tm} \\ F \downarrow & & \downarrow F \\ \mathcal{L} & \xrightarrow{\text{Id}} & \mathcal{L}_{tm} \end{array}$$

like in 3.4. Following 4.1, f is a trivial cofibration in \mathcal{K} if and only if Ff is a trivial cofibration. If the same holds for cofibrations then \mathcal{K} is left-induced in the sense of [11].

(2) In particular, the existence of pseudopullbacks would imply the existence of lax limits in **CMOD**. But Barwick proved that they always exist and are preserved not only by W_2 but also by W_1 ([4] 2.30).

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